COMPUTING EQUILIBRIUM PATHS OF CONTINUOUS FAMILIES OF ECONOMIES
A HOMOTOPY TECHNIQUE AND ALGORITHMS

By

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TO MY PARENTS

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This study is in two parts. The first part develops algorithms to trace equilibrium paths of continuously changing economies. The underlying technique -- to be called the *homotopy technique* -- appears to be of independent interest.

The homotopy technique suggested here is related to the homotopy principle of Eaves and others. In the context of studying continuously changing, parametric systems an idea similar to the homotopy principle is implemented with a different emphasis. When $F: M \times [0,1] \to R^n$, $M \subset R^n$ represents a one-parameter family of systems $(F(\cdot,t)$ for $0 \le t \le 1)$ a homotopy path $\phi: [0,1] \to M \times [0,1]$ is traced such that $\phi(t)$ approximates a solution of $F(\cdot,t)$ for all $t \in [0,1]$. Starting at a solution to the initial system $F(\cdot,0)$ a route of solutions is followed using complementary pivoting until it terminates with a solution to the final system $F(\cdot,1)$. The starting solution of $F(\cdot,0)$ can be computed using a first phase application of the homotopy principle.

The homotopy technique thus implemented in two phases can be regarded as the homotopy principle with a path constraint in the following sense. The path initiated at the trivial solution is required to join, at an intermediate stage, the solution path of the given family of systems to be computed and follow it to termination.

The homotopy technique is applicable to a wide-ranging set of parametric problems in optimization, complementarity, systems of equations, game and economic equilibria and fixed point computations. But in this work the focus is on equilibrium paths of economies.

One of the algorithms developed can be viewed as a parametric version of Scarf's algorithm in the sense of the homotopy technique. A new kind of nonsimplicial subdivision is constructed; its cells are 'pseudocones' in $R^n_+ \times [0,1]$ generated by an a priori chosen subdivision of $S \times [0,1]$ of mesh δ. The degree of approximation obtained is in terms of δ. This seems to be an advantage over more plausible finite subdivisions of $R^n_+ \times [0,1]$. A refining version of Scarf's algorithm is obtained as a by-product.

Three main algorithms are developed, one for each of the three categories of economies considered, a neoclassical model of exchange, a general Walrasian model with production and more elaborate economies characterized by point-to-set maps. Each of the algorithms traces an approximate general equilibrium path as the initial economy is continuously deformed into the final one. The basic algorithm can also be adapted for different scenarios like (i) only the initial and final economies are specified, (ii) the initial economy and a projected path of evolution are given. We obtain the homotopy principle for computing unknown equilibria of a given economy by taking the initial economy in (i) to be a trivial one.

The degree of approximation obtained in any particular implementation is estimated in relation to δ the mesh of the relevant subdivision. The approximate path converges uniformly to the actual path as $\delta \to 0$

Using suitable adaptations of the main algorithms a new dynamic framework is formulated in Part II for the analysis of various problems in economic policy evaluation, public goods economics, international trade, urban economics and spatial economics. Incorporation of dynamic policy alternatives and various parametric changes in the economy into our framework is indicated. All prior work in these areas using computed general equilibria has been done in a comparative statics setting, but the need for a dynamic framework had been emphasized.

PART I

THE HOMOTOPY TECHNIQUE AND ALGORITHMS

CHAPTER 1

INTRODUCTION AND SUMMARY OF RESULTS

1.1 Introduction and Overview of Results

This study is in two parts. The first part centers on the task of formulating algorithms to trace paths of approximate equilibria of general equilibrium models of the economy under continuous deformation. The method underlying these algorithms -- to be called the homotopy technique -- which provides as a specific case a parametric version of the Scarf algorithm [107] seems to be of independent interest. It has wide-ranging applications to parametric versions of problems in complementarity, optimization, fixed point computations, game and economic equilibria, numerical analysis, engineering, etc.

The second part deals with some economic applications of the algorithms of Part 1. Problems in evaluation of tax policies, international trade, public goods economics and urban economics are given a new dynamic framework for analysis using paths of general equilibria of the underlying systems.

1.1.1 Homotopy technique

A well-known technique for studying systems, say $F: M \subset E^n \to R^n$ (for example, finding fixed points, finding zeroes, etc.) is to imbed f into a one-parameter family of systems, a homotopy $F: M \times [0,1] \to R^n$ where $F(\cdot,0)$ is a trivial system with a known solution and $F(\cdot,1) = f$ is the system under study. Then a homotopy path is traced starting

from the known solution and terminating in a solution of $F(\cdot,1)$. If F does not satisfy differentiability conditions the strategy of tracing the homotopy path is to track solutions to a piecewise affine approximation H of F with respect to a subdivision of $M \times [0,1]$ (i.e., a subdivision of $M \times [0,1]$ into $(n+1)$-dimensional closed convex polyhedra such that the restriction of H to each polyhedron is affine). There are many algorithms which follow such paths [26, 30–32, 34–35, 45, 67, 91] on triangulations of $M \times [0,1]$ by complementary pivoting. The *homotopy principle* as presented by Eaves [32] unifies many of these algorithms. The emphasis of these algorithms and the homotopy principle has been that of starting from a trivial system and terminating with a solution to the system of interest. However, in the context of studying continuously changing (parametric) systems, essentially a similar idea can be implemented with a different emphasis.

When $F: M \times [0,1] \to R^n$ represents a one-parameter family of systems (i.e., $F(\cdot,t)$ for $0 \le t \le 1$) a homotopy path $\phi: [0,1] \to M \times [0,1]$ can be traced such that $\phi(t)$ approximates a solution of $F(\cdot,t)$ for $0 \le t \le 1$. A route of solutions starting at a solution to the initial system $F(\cdot,0)$ is followed until it terminates with a solution to the final system $F(\cdot,1)$. The starting solution of $F(\cdot,0)$ can be computed using a first phase application of the homotopy principle, if necessary.

The adaptation we have made will be important from the point of view of applications to study parametric systems. For convenience of discussion, the adapted version will be called the homotopy technique. Though at first glance there seems to be no essential mathematical difference between the homotopy principle and the homotopy technique

suggested here there is one crucial difference. The homotopy technique as implemented above can be thought of as the homotopy principle with a path constraint in the following sense. The path starting from the trivial solution is required to join, at an intermediate stage, the solution path of the given family of systems (to be computed) and follow it until termination.

The homotopy technique is applicable to parametric versions of problems in various areas: fixed point computation, solving systems of equations, complementarity, optimization, mechanics, engineering, numerical analysis, game and economic equilibria, etc. But we have limited the scope of this work to its applications in the specific context of computing equilibrium paths of economies under continuous deformation.

Three algorithms are developed in Chapter 5, one for each class of general equilibrium models discussed in Chapter 3. The first two algorithms compute paths of ε-approximate Brouwer and Kakutani fixed points respectively. The third algorithm required a distinctive construction. It can be viewed as a parametric version (in the sense of the homotopy technique) of Scarf's algorithm [107].

1.1.2 Parametric Scarf Algorithm

In order to implement the homotopy technique for this problem a new kind of a nonsimplicial subdivision has been constructed. The cells of this subdivision are "pseudocones" in $R_+^n \times [0,1]$ generated by an a priori chosen subdivision of $S \times [0,1]$ of a preselected mesh δ. The advantage of this type of subdivision over more plausible finite subdivisions of $R_+^n \times [0,1]$ (say cones) is that the mesh δ of the subdivision of $S \times [0,1]$ can be directly related to the degree of

approximation obtained. As a by-product of this technique a continuously refining version of Scarf's algorithm is obtained.

1.1.3 Approximating equilibrium paths

Chapter 3 describes three models of the economy of varying generality with corresponding characterizations of general equilibria. The first is an abstract model of an exchange economy. The second set of models are more elaborate economies where the defining functions are point-to-set maps and their equilibria are characterized by versions of the Kakutani fixed point theorem. The Walrasian general equilibrium model with production is treated as the third category. Chapters 4 and 5 develop the algorithms for computing approximate equilibrium paths of the above-mentioned classes of economies under continuous deformation.

The homotopy technique is presented in Chapter 4, it is contrasted with the homotopy principle. The underlying tools for the algorithms are collected together in this chapter.

Chapter 5 develops three main algorithms, one for each model of the economy described in Chapter 3.

Each algorithm traces an approximate equilibrium path as the initial economy is deformed into the final economy. If the dynamics of change between the initial and final economies are specified, then the algorithmic output approximates this temporal evolution. If the dynamics is unknown or unspecified, and only the initial and final economies are given, possible deformations can be constructed using *homotopy theory*. Then the algorithm approximates possible equilibrium paths between the initial and final economies. If the initial economy and an evolution path for it is given then the algorithm would provide an approximate path of evolution of the equilibria. As a by-product

of this technique we obtain an algorithm to compute equilibria of a given economy. This is the special case of the algorithm when the initial economy is taken to be an artificial economy (whose unique equilibrium is easily specified) and the final economy as the given economy whose equilibrium point is to be computed. Thus our algorithm reduces to the homotopy principle for computing economic equilibria.

Two kinds of results obtained formalize the sense of the approximation implied in the algorithms. The first is a limiting argument which asserts that as the mesh of the simplicial subdivision is reduced to zero the approximate equilibrium path converges to the actual one. The second relates the closeness of the approximation achieved to the size of the mesh of the simplicial subdivision on which the algorithm is implemented.

1.1.4 Economic applications

In Part II the algorithms of Chapter 3 are applied to study various problems in economics based on general equilibrium paths of the underlying systems subject to continuous deformations induced by the issue under study. A new dynamic framework has been devised to study dynamic paths of policy alternatives and economic reforms.

In Chapter 6, the algorithms are adopted for a dynamic analysis of various issues in tax policy evaluation and public goods economics. The tax structure allows *dynamic* modeling of various differential taxation regimes and resulting distortionary effects on the economy. Prior work in the area [5, 51-53, 108-115, 130] has used a comparative statics setting but has emphasized the need for a dynamic framework.

Chapter 7 develops a similar dynamic framework for analysing issues in international trade. The framework allows for a dynamic analysis of

international markets under continuous deformations induced by changes in tariff structure, import quotas, fiscal harmonization in common markets and other issues in world trade.

Chapter 8 focuses on spatial equilibrium models, especially those in urban economics. The effects of continuous changes in urban transportation, property taxes, congestion and other related issues can be analyzed using the dynamic setting developed here. Prior work in the area [4-5, 79-81] has used a static model. Arnott and MacKinnon [5, p. 404] identify the static setting as the most serious deficiency of the existing models.

We have also indicated an adaptation of our algorithms to compute general equilibrium paths of a general class of spatial equilibrium models under deformation.

The homotopy technique and the algorithms of this paper (especially the third) can be applied to a far wider range of problems than those specifically discussed in this paper. Prospective applications include parametric versions of various problems so far tackled using complementary pivot theory (see Section 1.4 and Remark 1.3.4).

1.2 The Economic Setting

One of the major concepts of Mathematical Economics during this century has been that of *general equilibrium*. Its major theme is that the behavior of a complex economic system can be viewed as an equilibrium arising from the interaction of a number of economic units with different motivations. The majority of economic decisions is seen as delegated to subordinate units whose behavior is elicited rather than controlled. A mature, functioning economic system engaged in the production of a variety of consumer goods and industrial products,

should generate a series of political, administrative and institutional signals that motivate and reward its agents and prompt them to select mutually consistent and feasible economic decisions.

The general equilibrium has been studied at different levels of generality [6, 11, 17-25, 41, 54-58, 74, 107-120, 122, 127]; the existence of such a state, its optimality, its robustness, its uniqueness and stability -- all these issues have been studied [6, 8, 11, 17, 18, 21, 23-25, 42, 109, 116-120, 122, 127], and continue to be studied in a variety of settings. The models whose equilibria are investigated are becoming more and more general -- they incorporate public goods [97-99], externalities of various kinds [11, 20, 108-115, 122, 130], differences in agents' information [6, 58, 122], etc.

The appeal of such results is easy to understand: the equilibrium studied may sustain an optimal resource allocation and accord well with individual incentives. Therefore it is important to extend the conditions under which an equilibrium exists, sustains its optimality, maintains its stability and so on.

The main underlying devices used in the study of the complex systems arising in general equilibrium models have been various *fixed point theorems* from Algebraic Topology. Brouwer's fixed point theorem (see 2.4.1) and Kakutani's fixed point theorem [59] with their variations and extensions have provided proofs of existence of solutions to various nonclassical general equilibrium models of economics during the last three decades [6-8, 11, 17-25, 54-57, 128]. But it was only in the last decade that these existence results of general equilibrium theory have been cast in a constructive (computational) setting by the so-called *fixed point algorithms*. The pioneering work of Scarf

[105-107] and supporting advancements made by various other researchers [12, 13, 26-40, 43-50, 60-73, 75-86, 89-96, 101-104, 116, 123-126, 131-132] has made it possible to compute the equilibria of fairly large-scale economies with reasonable effort. This possibility of obtaining computed values for general equilibria has opened up a new framework for studying numerous problems using models which incorporate all the complex interrelationships of general equilibrium theory [4-5, 46, 49, 54, 78-81, 97-99, 107-115]. Some of the areas studied in this framework have been:

(1.2.1) Analysis of taxation policies [5, 51-53, 108-113, 115],

(1.2.2) Impact of price distortions [114, 130],

(1.2.3) Problems in urban economics [4-5, 79-81],

(1.2.4) Public goods economies [97-99],

(1.2.5) Issues in international trade [46, 54, 114, 129-130] and

(1.2.6) A general class of spatial equilibrium models [80-81]

All these and similar other studies have used a *comparative statics* procedure, i.e., they have compared the equilibria in the situations before and after the change under the study was made. Clearly such a procedure is very limited in that it provides no information about the direction of change from the initial economy or what transpired in the interim. Ideally one would like to know which way a change in the economy would affect the equilibria, what would be the equilibrium path as the economy is gradually changed from the existing one to another, etc. The thrust of this paper is to devise algorithms which approximate equilibrium paths of changing economies with different characterizations of equilibria. Some applications of such algorithms to areas (1.2.1-6) mentioned earlier are also explored.

1.3 The Problem and the Approach

The research in this paper is directed at achieving three major objectives.

1.3.1 Algorithms

Let E(t) represent an economy for all t, $t_0 \leq t \leq t_1$, q(t) is an element of the equilibrium set of E(t). The problem studied here is to devise an algorithm to construct a connected path $\bar{q}(\cdot)$ of approximate equilibria of $E(\cdot)$, the sense of approximation to be made precise. The algorithms devised should be general in approach to subsume various characterizations of equilibria depending on the assumptions made and the level of generality incorporated in the model.

1.3.2 Homotopy technique

The second objective is to formulate a general technique to construct approximate solution paths of continuous families of systems arising in game and economic equilibria, complementarity, optimization, fixed point computations, systems of equations, etc. The technique -- to be called homotopy technique -- is an adaptation of the homotopy principle of Eaves [32] and others with a different emphasis in implementation. This technique underlies the algorithms of 1.3.1.

1.3.3 Applications

In the last eight years a number of economic problems have been analyzed, e.g., (1.2.1-6), based on computed solutions in a general equilibrium framework. The second objective of this work is to adapt the algorithms of (1.3.1) for specific application to some of these problems by providing a dynamic framework of continuous change as opposed to the existing one of comparative statics.

The first two objectives are discussed in Part I of this paper while the third is the topic of Part II.

1.3.4 Remark

Even though the algorithms of this paper are presented in the context of equilibria of economies under deformations, their prospective applications are quite wide-ranging and include *parametric* versions of various problems [1-3, 12-16, 26-35, 37-38, 43-50, 60-73, 75-86, 89-96, 103-115, 131] whose solutions have been attempted using *complementary pivot theory*.

1.4 Related Approaches

Our development of the *homotopy technique* and the algorithms of Part I are based on *complementary pivot theory*; in the underlying ideas and its presentation it is the closest to the *homotopy principle* used by Eaves in [32]. His use of PL Homotopies in [32] unifies various applications of a new and powerful technique called *complementary pivot theory* which has emerged over the last decade. The applications of this versatile technique in its various forms pervade an impressive range of topics including linear and nonlinear complementarity, fixed point theory, linear and nonlinear optimization, equilibria of games and economies, nonlinear systems of equations, saddle-point problems, boundary value problems, structural mechanics, geometry, plasticity analysis, network problems, pricing of utilities, urban transportation models and many others.

The beginnings of the excitement in the area could be traced to the papers of Lemke [75] and Lemke and Howson [77] where a new combinatorial scheme was presented to compute the equilibrium point of bimatrix games. Three years later in 1967 Scarf [105] used the same underlying priciple to approximate the Brouwer fixed points of maps thereby extending the purview of applications of this combinatorial

technique to nonlinear problems. The generality of the underlying combinatorial structure was recognized [26-27, 47] and its applications to various areas in applied mathematics and economics became substantial: Linear Complementarity [13, 16, 28-29, 43, 62, 68, 70, 75-77, 82-86, 95], Nonlinear Complementarity [26, 37, 39, 60-61, 66, 89-90, 94], Fixed Point Computations and Solving Nonlinear Systems of Equations [1-3, 10, 12, 27, 31-35, 40, 44, 48, 67, 71-73, 85, 92, 122-125, 131-132] have been probably the most active areas.

Eaves [30, 31], Eaves and Saigal [34] used the idea of PL Homotopies for computing fixed points, the *homotopy principle* was originally formulated in [26, 30] and presented as a unification of various applications of *complementary pivot theory* in a crystallized and elaborated form in [32]. Eaves [32] has had a major influence on this dissertation.

To follow the homotopy paths we will use simplicial pivoting as in [1-3, 12, 27, 30-35, 37, 39-40, 48, 67, 101] but other methods have been suggested and used. Continuation methods [64-65, 96, 128], Solving appropriate differential equation systems [45, 92], *Global* Newton methods [120] are some of the alternatives which can be used if the systems under study satisfy required differentiability conditions. In the economic models used in this paper no differentiability assumptions are made on the characterizing functions, hence we use simplicial pivoting to follow the equilibrium paths.

CHAPTER 2

PRELIMINARIES AND SOME USEFUL THEOREMS

2.1 Definitions and Notation

For convenience some basic definitions and conventions for notation and referencing are collected in this section

2.1.1 Hulls

By an *affine* combination of points $x_1, x_2, x_3, \ldots, x_p$ in some Euclidean space R^n we mean the point $\Sigma \lambda_i x_i$ where $\Sigma \lambda_i = 1$, $\lambda_i \in R$, by a *convex* combination, we mean $\Sigma \lambda_i x_i$ where $\Sigma \lambda_i = 1$, $\lambda_i \geq 0$. An affine (convex) set is one closed under affine (convex) combinations. Given a set C then *aff*(C) (*conv*(C)) is the smallest affine (convex) set containing C. Thus aff(C) is the affine hull of C, and conv(C) is the convex hull of C

2.1.2 Abbreviations:

The following further abbreviations will be used:

bd C = boundary of C,

cl C = closure of C

int C = interior of C

dim C = dimension of C = dim {aff(C) - C}.

2.1.3 Vectors and Matrices:

As a rule, the lower case letters u, v, x, y, etc., denote vectors in R^n or R^{n+1}. Points in $R^n \times [0,1]$ might be represented as $u = (x,t)$ where t is the $(n + 1)$-st coordinate of u. For $x \in R^n$, x^i $(i = 1, 2, \ldots, n)$ denotes the i-th coordinate of x

If $A \in R^{m \times n}$ is a real $m \times n$ matrix, we denote its submatrices using the following conventions. A^i denotes the i-th row of A and A_j its j-th column. If $\gamma \subset \{1, 2, \ldots, m\}$, A^γ denotes the submatrix formed by the rows indexed by γ in their natural order. Similarly if $\beta \subset \{1, 2, \ldots, n\}$ then A_β represents the submatrix formed by the respective columns of A in their natural order.

e_i denotes i-th unit vector in R^n. R^n_+ is the nonnegative orthant of R^n, i.e., $\{x \in R^n \colon x^i \geq 0, i = 1, 2, \ldots, n\}$. S denotes the standard simplex in R^n, i.e., $S = \text{conv}\{e_i, i = 1, 2, \ldots, n\}$. A^T denotes the transpose of matrix A.

2.1.4 Set Operations

The usual set theoretic operations of union, intersection and difference will be denoted by $\cup$, $\cap$ and $\sim$ respectively.

For $C, D \subset R^n$,

$$C + D = \{x + y \in R^n, x \in C, y \in D\}$$

$$C - D = \{x - y \in R^n; x \in C, y \in D\}.$$

For $\lambda \in R$,

$$\lambda C = \{\lambda x \colon x \in C\}$$

For $C \subset R^n$ and $D \subset R^m$,

$$C \times D = \{(x,y) \in R^{n+m}; x \in C, y \in D\}.$$

2.1.5 Norms and meshes.

$\| u \| = [u^T u]^{\frac{1}{2}}$ denotes the Euclidian norm of the vector $u \in R^n$

$|u| = \ell\infty$-norm of $u \in R^n$, i.e., $\max\limits_{1 \leq i \leq n} |u^i|$

$B(x,\rho)$ = Open ball with center x, radius ρ,
i.e., $\{y \in R^n \colon \| x - y \| < \rho\}$.

$B(C,\rho)$ = ρ-neighborhood of $C = C + \rho B(0,1)$.

diam C = Diameter of $C = \sup\{\| x - y \| \colon x, y \in C\}$

[illegible] $(G \in G)$
where G is a family of subsets of R^u.

2.1.6 Ordering

For $u, v \in R^n$, $u \geq v$ means $u^i \geq v^i$ for $i = 1, 2, \ldots, n$. $u > v$ means $u^i > v^i$ for $i = 1, 2, \ldots, n$.

u is said to be *lexicographically less* than v (denoted as $u \leq^{\ell} v$) if either $u = v$ or $u^1 = v^1, \ldots, u^{j-1} = v^{j-1}$, $u^j < v^j$ for some $j = 1, 2, \ldots, n$.

In Chapter 3 the preference relation $\gtrsim_i$ for i-th consumer is defined on R^u_+ by $x \gtrsim_i y$ iff x is at least as much preferred as y by consumer i.

2.1.7 Referencing

All items including equations in a given section are numbered consecutively. Item i.j.k refers to item k in Section j of Chapter i (i.j.k) refers to Equation (i.j.k). Section i.j refers to Section j of Chapter i.

2.2 Properties of Upper Semi-continuous Point-to-set Mappings

In the following discussion, denote by $P(R^n)$ the set of all subsets of R^n. Let $C \subset R^m$ and $H: C \to P(R^n)$ be a point-to-set mapping.

2.2.1 Definition:

H (as given above) is *upper semi-continuous* (U.S.C) if

(i) for all $x \in C$, $H(x)$ is compact, and

(ii) for all $x \in C$, for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $y \in B(x,\delta) \cap C$ then $H(y) \subset B(H(x),\varepsilon)$.

If $h: C \to R^n$ is a function then $H: C \to P(R^n)$ with $H(x) = \{h(x)\}$ is a point-to-set mapping -- call it $\{h\}$. Note that h is continuous iff $\{h\}$ is U.S.C.

A very important property of U.S.C. mappings is the following

2.2.2 Theorem

If $H: C \to P(R^n)$ is U.S.C., $\{x^\ell\}$ is a sequence of points in C such that $x^\ell \to x^*$,

$y^\ell \in H(x^\ell)$ for all ℓ,

and $y^\ell \to y^* \in R^n$, then $y^* \in H(x^*)$.

Proof. For any $\varepsilon > 0$, there is a q such that $\| y^* - y^q \| \leq \frac{\varepsilon}{2}$ and $H(x^q) \subset B(H(x^*), \varepsilon/2)$. Thus $y^* \in B(H(x^*), \varepsilon)$ for all $\varepsilon > 0$. Since $H(x^*)$ is compact, $y^* \in H(x^*)$.

Two additional properties are stated without proof. See Berg [9, p. 116-120] or Merrill [91, p. 17-19] for details.

2.2.3 Theorem

If H is a U.S.C. point-to-set mapping, then $H(D)$ is compact for any compact subset $D \subset C$. $[H(D) = \underset{x \in D}{\cup} H(x)]$

2.2.4 Theorem

If H is U.S.C., then $G: C \to P(R^n)$ defined by $G(x) = \operatorname{conv}(H(x))$ is an U.S.C. point-to-set mapping.

2.3 Some Ideas from Homotopy Theory

When we consider continuous changes over time on a model of the economy, closely related are questions whether one map can be continuously transformed into another. The elaboration of such ideas forms part of Homotopy Theory [88, 121].

Let X and Y be topological spaces. Two continuous maps $f, g: X \to Y$ are said to be homotopic if f can be continuously deformed into g, i.e., if there exists a continuous family of maps $f_t: X \to Y$, $(0 \leq t \leq 1)$ such that $f_0 = f$ and $f_1 = g$. Now to characterize a

continuous family of maps $f_t : X \to Y$, we can consider a single map $F: X \times [0,1] \to Y$ defined by the rule $F(x,t) = f_t(x)$ $(x \in X, t \in [0,1])$. When we say that the maps f_t form a continuous family of maps we merely mean that F is a continuous map of $X \times [0,1]$ to Y.

2.3.1 Definition.

Two continuous maps $f, g: X \to Y$ are *homotopic* (or f is homotopic to g) if there exists a continuous map $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. The map F is said to be a *homotopy* and we write $f \simeq g$ (f is homotopic to g).

The following theorem gives a very useful result.

2.3.2 Theorem.

Let $f, g: X \to Y$ be two maps. If for each $x \in X$, $f(x)$ and $g(x)$ can be joined by a straight line segment in Y, then $f \simeq g$.

Proof. Indeed the convex combinations of f, g furnish a convenient homotopy. For details, see Maunder [88, p. 26].

Thus, we have, that any two maps $f, g: X \to R^n$ must be homotopic. An analogous result can be shown to hold for point-to-set maps into convex subsets of R^n using similar arguments. Since the maps characterizing the economic models often are those into R^n or its convex subsets, Theorem 2.3.2 guarantees the existence of a homotopy to deform the initial economy into the final one. Such a homotopy can be used (when the dynamics of change from the initial economy to the final one is unspecified) to define possible deformations between the economies. The trajectory of fixed points of continuous functions under homotopic deformations is an important issue related to the algorithms of this paper; it will be discussed in the following section. Piecewise affine homotopies which play an important role in the algorithmic machinery will be characterized and studied in Chapter 4.

2.4 Some Underlying Fixed Point Theorems

Some fixed point theorems, useful in characterizing equilibria of economies or equilibrium paths of economies under deformation are given below. These theorems are stated in a manner best suited to the applications in this paper (and not necessarily in their most general form). The proofs of these theorems are not included here. References are indicated; moreover the algorithms of Chapter 5 provide constructive proofs to all these theorems.

2.4.1 Theorem (Brouwer).

A continuous mapping $f: S \to S$ (S is the standard simplex) has a fixed point, i.e., $x \in S$ such that $x = f(x)$.

Equilibria of the Walrasian exchange economies, e.g., [6, p. 26] are characterized using this theorem. Many of the numerous articles dealing with fixed point algorithms contain a constructive proof [27, 30, 39, 63-65, 71, 73, 91, 105, 107, 124, 132].

2.4.2 Theorem (Kakutani [59]).

$F: S \to C(S)$ is a U.S.C. point-to-set mapping. $C(S)$ is the collection of nonempty, closed, convex subsets of S. Then there exists a fixed point, i.e., $x \in S$ such that $x \in F(x)$.

Equilibria of a general Walrasian model with production [123, p. 65] and more elaborate models of the economy [6, 11, 17, 22, 46, 78, 97, 122] can be characterized using Kakutani fixed points. Constructive proofs of the theorem can be found in many sources [27, 30, 32, 35, 39, 40, 63, 91, 107].

Of direct relevance for this work are the parametric versions of the above theorems. A special case of a theorem of Browder [10, p. 186] will be of interest.

2.4.3 Theorem

Let $F: S \times [0,1] \to S$ be continuous. Then there exists a connected set C in $S \times [0,1]$ meeting $S \times \{0\}$ and $S \times \{1\}$ such that $F(x,t) = x$ for each $(x,t) \in C$.

Constructive proofs of this theorem are given in Eaves [26, 32] and Freidenfelds [39, 40]. Freidenfelds [40] has generalized the above theorem to a parametric version of Theorem 2.4.2 and provided a constructive proof using *complementary pivot theory*.

2.4.4 Theorem

Let $F: S \times [0,1] \to C(S)$ be a U.S.C. point-to-set map and $C(S)$ as in 2.4.2. Then there exists a closed connected set $C \subset S \times [0,1]$ which meets both $S \times \{0\}$ and $S \times \{1\}$ and $x \in F(x,t)$ for all $(x,t) \in C$.

Theorems 2.4.3 and 2.4.4 provide a motivation for the *homotopy technique* used in the algorithms of Chapter 5.

Mas-Colell [87] has extended these theorems further enlarging the possible scope of a homotopy technique. But these extensions will not be used in chapters to follow.

CHAPTER 3

THE ECONOMIC FRAMEWORK

3.1 Introduction

To characterize the competitive equilibria and equilibrium paths of economies we need to specify the models of the economies used and characterize their equilibria. Based on the theorems used to characterize equilibria, three representative models of the economy with varying levels of generality are discussed below. The first is an abstract model of an exchange economy whose equilibria can be characterized using Brouwer's fixed points. The second set of models are more elaborate economies where the defining functions are point-to-set maps and their equilibria are characterized using different versions of the Kakutani fixed point theorem. A combinatorial theorem of Scarf [107, p. 76] (to be referred to as Scarf's theorem in the rest of this paper) can be used to construct the equilibrium configurations of a third model -- the Walrasian general equilibrium model with production. In Chapter 5, three algorithms are devised to trace the equilibrium paths of the above three models of economies under deformation.

3.2 Abstract Model of an Exchange Economy

In this section we consider an abstract model of an exchange economy where production will be disregarded. There are finitely many agents, indexed by $i = 1, 2, \ldots, n$, all dealing with a finite set of commodities, $\ell = 1, 2, \ldots, m$.

A commodity is a good or service completely specified physically, temporally and spatially. It is assumed that they are perfectly divisible, i.e., the quantity of any of them can be any real number. The commodity space can be represented as R^n, by stipulating the convention that quantities made available to (by) an economic agent, i.e., inputs (outputs) are represented by nonnegative (nonpositive) real numbers. The existence of all futures markets will also be assumed.

3.2.1 Preference relation

Consumer i has an initial endowment of the commodities, $w_i \in R^n_+$. He also has a preference relation $\succsim_i$ which dictates his decision when confronted with the choice between two commodity bundles say x and y, if $x \succsim_i y$ then x is at least as much preferred as y. Now $\succsim_i$ is a binary relation defined on R^n_+ which is *reflexive*, *transitive* and *complete*. It may not be *antisymmetric*. Define $x \succ_i y$ iff $x \succsim_i y$ and not $y \succsim_i x$. To express the desirability of all commodities the following assumption is made.

3.2.2 Strong monotony assumption

If $x \geqq y$, and $x \neq y$, then $x \succ_i y$.

A price system is a nonzero vector $p \in R^n_+$ where (if $p^\ell > 0$) p^k/p^ℓ units of commodity ℓ are necessary to purchase one unit of commodity k. Given such a price system p, the budget set of consumer i is $B_i(p) = \{x \in R^n_+ : p^T(x - w_i) \leq 0\}$. If $p > 0$, $B_i(p)$ is compact. Clearly the budget set is not affected if the price system is multiplied by a positive scalar. Therefore it is enough to consider price vectors in the standard simplex $S = \{p : \Sigma p^\ell = 1,\ p \geqq 0\}$.

If the preferences further satisfy the condition that for all $z \in R_+^n$, the sets $\{x \in R_+^n : z \gtrsim_i x\}$ and $\{x \in R_+^n : x \gtrsim_i z\}$ are closed, then there exists a continuous utility function $u^i : R_+^n \to R$ such that $u^i(x) \geq u^i(z)$ iff $x \gtrsim_i z$, see Debreu [17, p. 54]. Such preferences are called continuous preferences. When $\gtrsim_i$ is continuous then the agent i will choose an element of the demand set

$$d_i(p) = \{x \in R_+^n \mid x \text{ is } \gtrsim_i \text{-maximal in } B_i(p)\}.$$

Because of the strong monotony of preferences (in 3.2.2) we have the identity $p^T d_i(p) = p^T w_i$ for all $p \in S$. Further if the following convexity assumption is made, $d_i(p)$ is a singleton.

3.2.3 <u>Assumption</u>

If $x \geq y$, $x \neq y$ then $tx + (1 - t)y \succ_i y$ for all $t \in (0,1]$

Under the assumptions made, the demand function, $d_i : S \to R_+^n$ is continuous [6, 17]. Set $d(p) = \sum_{i=1}^{m} d_i(p)$, $w = \sum_{i=1}^{m} w_i$. $d(p)$ is the aggregate demand function and w the total initial endowment of the economy. We have, $p^T d(p) = p^T w$. Let $g(p) = d(p) - w$ be the excess demand function. Then $p^T g(p) = 0$, i.e., the value of excess demand is zero for all $p \in S$. This equality, called the *Walras law* is a basic relationship of Walrasian economics [127]. Since each d_i is continuous, $g : S \to R^n$ is continuous. This exchange economy with m agents can be characterized by an m-tuple $(\gtrsim_i, w_i)$ or by the more derived specification (d_i, w_i). On an aggregate level the economy can be specified as (d, w).

Now an equilibrating price system has to bring into harmony the disparate actions of all the agents (consumers). At such a price system $\bar{p}$ the excess demand $g(\bar{p}) \leq 0$.

3.2.4 Definition

$p \in S$ is an equilibrium price vector if $g(p) \leq 0$.

Note that if Walras law holds and $\bar{p}$ is an equilibrium price vector then $g^{\ell}(\bar{p}) \leq 0$ for all $\ell = 1, 2, \ldots, n$ with equality if $\bar{p}^{\ell} > 0$. Thus all markets are in balance except possibly those of free goods, which can be in excess supply.

The existence of such an equilibrium can be proved by using Brouwer's fixed point theorem (2.4.1) on a suitable map $f: S \to S$ as constructed such that its fixed points $\bar{p}$ coincide with the equilibrium points of the economy. Details of such a construction are deferred till 5.2. Also see Scarf [107, p. 30], Arrow and Hahn [6, p. 27] or Todd [123, p. 13].

Now when the initial economy $(d(t_0), w(t_0))$ at time t_0 is gradually changed according to some dynamics to the final economy $(d(t_1), w(t_1))$ at time t_1, the algorithm devised in 5.2 will trace the equilibrium path of the economy under deformation.

3.3 More Elaborate Economies

In the simple model considered above we made various assumptions so that the economy could be characterized in terms of excess demand functions $g: S \to \mathbb{R}^n$. Many of these restrictions could be relaxed and the model made more elaborate by incorporating details of production and issues of welfare. For example, see Englehl [96], Debreu [17, p. 37-88] or Arrow and Hahn [6, p. 52-125]. In these models the relations characterizing the economies are point-to-set maps (also called correspondences) which are often U.S.C. (recall 2.2). The main device behind the existence of equilibria in these elaborate

economics has been the Kakutani fixed point theorem and its variations applied to an appropriate U.S.C. point-to-set map $f: S \to C(S)$, where S is the standard simplex in some Euclidian space and $C(S)$ the family of compact convex subsets of S.

3.3.1 Remark.

In these formulations, S need not be the price simplex as in 3.2, it could be a simplex of utility weights, as in Negishi [96] or a simplex of normalized vectors of utility levels of all the agents in the economy, as in Arrow and Hahn [6] or some similar construction. In some cases the original economy characterized by maps defined on an arbitrary compact set C might be shifted onto the standard simplex S by imbedding C in S. To reflect this difference from 3.2 we will denote the argument of $f: S \to C(S)$ by x instead of p.

The map f is so constructed such that its fixed points $\{\bar{x} \mid \bar{x} \in f(\bar{x})\}$ correspond to the equilibrium points of the economy. These economies are so varied and numerous that they will not be described in any further detail. For our purposes the functions $f: S \to C(S)$ shall characterize these economies. In this framework, to trace the equilibrium path of these economies we need to follow the fixed points of the maps $f(\cdot,t): S \to C(S)$ as t goes from its initial value t_0 to its final value t_1. Our algorithm of 5.3 will be able to approximate such paths. Theorem 2.4.4 provides a motivation for this algorithm.

3.4 Walrasian Economies with Production

Now we discuss the general Walrasian model obtained by adding an explicit production sector to the exchange model of 3.2. As before

each consumer will determine his consumption plan by maximizing utility, subject to the constraint that expenditure at the prevailing set of prices shall not exceed the income generated by the sale of his productive factors. On the other hand, the independent producing units in the economy select production plans that maximize profits, all inputs and outputs being evaluated at the prevailing prices. The price system in the economy forms the link between these independent units of the economy, these prices must be such that they lead to mutually consistent decisions. The equilibrating prices must be such that no producer has a compelling motivation to search for higher profits by the selection of an alternate mode of production. On the other hand at these prices supply and demand should balance providing little inducement for any revision in the price system.

As in 3-2 w_i represents the vector of resources of the i-th consumer and $d_i(p)$ his demand at prices p, i.e., $d_i(\cdot)$ is the demand function. The third basic component of the economic process is the technology: the possibilities of production, described by an activity analysis matrix

$$A = \begin{bmatrix} -1 & 0 & \cdots & 0 & a_{1,n+1} & \cdots & a_{1,k} \\ & 1 & & 0 & & & \vdots \\ \vdots & \vdots & & & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & a_{n,n+1} & \cdots & a_{n,k} \end{bmatrix}$$

each of whose columns represents an activity, if activity j is used at level 1, $|A_j^\ell|$ units of commodity ℓ are supplied as output (if $A_j^\ell \geq 0$) and required as input (if $A_j^\ell < 0$). Note that the first n columns

correspond to an assumption of free disposal. The activities can be operated simultaneously, each at an arbitrary nonnegative level, so that the set of production plans available to the economy is given by the set $\{Ay: y \geq 0\}$. The model can be generalized to allow for more general production sets, if the customary convexity assumptions are made; see for example [17, p. 39]. We make the following assumption.

3.4.1 Assumption:

The set of activity levels that give rise to a nonnegative net supply of all commodities is a bounded set, i.e., $\{y| y \geq 0, Ay+w \geq 0\}$ is bounded.

Aggregating the individual demand functions $d^i(\cdot)$ we obtain $d(\cdot)$ the market demand function, similarly, w the vector of total resource endowment of the economy prior to production. $d: S \to R^n$ is continuous on S, as before. Walras law also holds, $p^T d(p) = p^T w$ at all prices p.

3.4.2 Definition

A competitive equilibrium for this economy is defined by a vector of prices $\bar{p} \in S$ and a set of nonnegative activity levels $\bar{y} \in R^k_+$ such that

(a) supply equals demand in all markets, i.e.,

$d(\bar{p}) = A\bar{y} + w$, and

(b) $\bar{p}^T A \leq 0$

From (a), $\bar{p}^T w + \bar{p}^T A\bar{y} = \bar{p}^T w$, and Walras law, we have $\bar{p}^T A\bar{y} = 0$. Then (b) implies $\bar{p}^T A_j \bar{y}^j = 0$ for $j = 1, 2, \ldots, k$. $\bar{p}^T A_j = 0$ when $\bar{y}^j > 0$. The interpretation is that no activity makes a positive profit at the prices $\bar{p}$, those activities used (at a positive level) must make zero profit. In particular, if a *free disposal activity* $(1 \leq j \leq n)$ is used, the price of the corresponding commodity is zero. In this sense, at the equilibrium, production is consistent with profit maximization.

A proof of the existence of equilibrium for this model can also be based on versions of Kakutani fixed point theorem (see Debreu [17, p. 83], Scarf [107, p. 120] or Todd [123, p. 65]). Scarf's algorithm of [105 or 107] based on Scarf's theorem provided the first constructive proof of the existence of equilibria for this economy. The algorithm devised in 5.4 also would provide such a proof merely as a by-product.

The economy described here can be characterized by the triplet (d,w,A). A continuous changing economy can be represented as a function of the parameter t, for $t_0 \le t \le t_1$, where $(d(\cdot,t), w(t), A(t))$ characterizes the economy at time t. The third algorithm we have devised, in 5.4, will trace an approximate equilibrium path, $(\bar{p},\bar{y}): [t_0,t_1] \to S \times R^k_+$ such that $(\bar{p}(t),\bar{y}(t))$ approximates the equilibrium prices and activity levels of the economy $(d(\cdot,t),w(t),A(t))$ at time t.

The model of this section provides the best framework for the applications of Chapters 6 and 7.

CHAPTER 4

BUILDING BLOCKS OF THE ALGORITHMS

4.1 Homotopy Technique

The underlying machinery of the algorithm -- which we call the *homotopy technique* is as follows. The changing system under study is formulated as a one-parameter family of systems -- a homotopy which represents the actual deformation, and at the initial and final levels corresponds to the initial and final system respectively. Then, starting from a solution of the initial system, a homotopy path of solutions is traced which terminates in a solution of the final system. When no differentiability assumptions are made on the systems under study, the method of tracing the homotopy path involves tracking solutions to piecewise affine approximations to the system at each level by the method of *complementary pivoting*. In this sense the homotopy technique is closely based on the homotopy principle of Eaves [32]. In the special case when the initial system considered is a convenient trivial system homotopic to the actual system under study then the homotopy technique reduces to the homotopy principle.

Let us make the distinction precise. A well known technique for studying systems, say $f: M \subset R^n \to R^n$ (for example, finding fixed points, zeros, etc.) is to imbed f into a one-parameter family of systems, a homotopy $F: M \times [0,1] \to R^n$ where $F(\cdot,0)$ is a trivial system with a known solution and $F(\cdot,1) = f$ is the system of interest.

Then a homotopy path is traced from the known solution terminating in a solution of $F(\cdot,1) = f$. When F does not satisfy differentiability conditions, the strategy of tracing the homotopy path is to track solutions to piecewise affine approximation H of F with respect to a subdivision of $M \times [0,1]$ (i.e., a subdivision of $M \times [0,1]$ into (n+1)-dimensional closed convex polyhedra such that the restriction of H to each polyhedron is affine). There are many algorithms which follow such paths on triangulations of $M \times [0,1]$ by complementary pivoting. The *homotopy principle* as presented by Eaves [32] unifies many of these algorithms. The emphasis of the homotopy principle has been that of starting from a trivial system and terminating with a solution to the system of interest. However in the context of studying continuously changing systems more or less the same principle can be implemented with a different emphasis.

When $F\colon M \times [0,1] \to R^n$ represents a one-parameter family of systems $(F(\cdot,t)$ for $t \in [0,1])$ a homotopy path $\phi\colon [0,1] \to M \times [0,1]$ can be traced such that $\phi(t)$ approximates a solution of $F(\cdot,t)$ for $t \in [0,1]$. The starting solution can be computed using a first phase application of the homotopy principle.

The adaptation we have made of the homotopy principle to formulate the homotopy technique will be very important from the point of view of studying parametric systems. Though at first glance there seems to be no essential difference between the homotopy principle and the homotopy technique suggested here there is one crucial difference. The homotopy technique as implemented in two phases can be thought of as the homotopy principle with a path constraint in the following sense. The path starting from a trivial solution is required to join, at an intermediate

stage, a solution path of the given system (which is to be computed) and stay with it until termination. Thus the building blocks of the homotopy principle are applicable to the homotopy technique, we will follow [32] for a presentation of the underlying tools.

The homotopy technique can be thought of as a generalization of the homotopy principle, and it is applicable in a very broad context to study systems under deformations and to solve parametric versions of various problems to which *complementary pivoting* has been applied over the last decade. The work of Eaves [30-32], especially [32], has had a major influence on the homotopy technique and the algorithms of Chapter 5.

In spite of the broad applicability of the homotopy technique we limit its scope in this paper to equilibrium paths of changing economies, i.e., the systems we study are various models of the economy, and the solutions of interest are their general equilibria. Consequently the algorithms of the next chapter are developed in the specific context of equilibrium paths of economies in a form more appropriate for the applications of Part II. But the remarks above and the detailed discussion in this chapter of the underlying general framework would enable the reader to see the algorithms in a much broader context.

In the following description of the building blocks of our algorithms we follow the presentation of [32] very closely.

4.2 Cells and Manifolds

In the context of our path-following algorithms we will use closed convex polyhedra as components of subdivision of the sets we work with

In this context, after Savea [32], we will call them *cells*. Although the basic properties are well known, we include them here for fixing the terminology and easy referencing. See also [100].

A *cell* in R^n is the convex hull of a finite number of points and half-lines (half-lines being sets of the type $\{x: x = a + tb,\ t \geq 0\}$ where a and b are fixed vectors in R^n). Thus, a closed convex polyhedral set, bounded or not, is a cell.

By an *s-cell* we mean a cell of dimension n. If an n-cell is the convex hull of $(n + 1)$ points we call it an *n-simplex*.

The interior int σ and boundary bd σ of a cell σ are the interior and boundary relative to aff(σ).

Let τ be a subset of an n-cell σ. If $x, y \in \sigma$, $0 < \mu < 1$, and $(1 - \mu)x + \mu y \in \tau$ imply that $x \in \tau$ and $y \in \tau$ then τ is called a *face* of a cell σ. Faces that are $(n - 1)$-cells are called *facets* of the cell, and those that are 0-cells are called *vertices* of the cell.

The intersection of a line $\{x + \mu y;\ \mu \in R\}$ and a cell σ is called a *chord* of σ. A nonempty chord may be of dimension 0 or 1. By a *ray* of σ we mean a subset of a chord of σ which is a half-line.

4.2.1 <u>Definition</u>

Let $\mathcal{G} \neq \phi$ be a finite or countably infinite collection of n-cells in R^n. Let $\mathcal{G}^i$ be the set of i-faces of the elements of $\mathcal{G}$. $G = \bigcup_{\sigma\in\mathcal{G}} \sigma$. We call, $(G,\mathcal{G})$ a *subdivided n-manifold* if

(i) Any two n-cells that meet do so in a common face.

(ii) Each $(n - 1)$-face of a cell lies in at most two n-cells.

(iii) Each x in G has a neighborhood meeting only a finite number of n-cells in $\mathcal{G}$.

For a given G, if there is a $\mathcal{G}$, such that $(G,\mathcal{G})$ is a subdivided n-manifold we call G an n-manifold.

4.2.2 Definition:

For (G,G) as in 4.2.1 if each n-cell of the subdivision G is an n-simplex we say that G *triangulates* G.

Of particular importance to us in constructing equilibrium paths are 1-manifolds. If a 1-manifold is connected we shall call it a *curve*. It is trivial to prove

4.2.3 Lemma:

A connected 1-manifold (i.e., a curve) is homeomorphic to either a circle or an interval. For the two cases above we call the curve a *loop* or a *route* respectively.

4.2.4 Lemma:

A 1-manifold is a disjoint collection of routes and loops.

4.2.5 Lemma:

A loop is compact. Thus a loop contains no rays, and if P subdivides a loop P, it is finite.

If γ is a ray of $\sigma \in G$, then we call γ a ray of (G,G).

4.2.6 Definition:

Let G be an n-manifold subdivided by G. By the *boundary* of G, ∂G, we mean the union of all $(n-1)$-cells of G which lie in exactly one n-cell of G.

A useful property of 1-manifolds from the algorithmic point of view follows, as in Eaves [32, p. 85].

4.2.7 Lemma:

If the subdivision of a 1-manifold is finite, the number of boundary points plus the number of rays of the 1-manifold is even.

4.2.8 Definition

Let G be an n-manifold and P a 1-manifold contained in G. If P is closed in G and $\mathrm{bd}\, P = P \cap \mathrm{bd}\, G$ we say that P is *neat* in G.

Let P be a 1-manifold neat in G and let $\mathcal{G}$ be a subdivision of G. Let $\mathcal{P}$ be the set of 1-chords of $\mathcal{G}$ of the form $P \cap \sigma$, where σ is an n-cell of $\mathcal{G}$. If $\mathcal{P}$ subdivides P we say that P is *neat in* $(G,\mathcal{G})$ or neat in the subdivided manifold G.

4.3 Piecewise Affine Maps

In this section the important concepts of piecewise affine maps and piecewise affine approximations are discussed.

4.3.1 Definition

Let $(G,\mathcal{G})$ and $(J,\mathcal{J})$ be subdivided manifolds. Let $F: G \to J$ be a continuous map which is *affine*, $(F(\mu x + (1 - \mu)y = \mu F(x) + (1 - \mu)F(y))$ on each cell of $\mathcal{G}$ and which carries each cell of $\mathcal{G}$ into a cell of $\mathcal{J}$. Such a map F is called *piecewise affine map*.

Given a cell $\sigma \in \mathcal{G}$ and $\tau \in \mathcal{J}$ with $F(\sigma) \subset \tau$ define $F_\sigma: \mathrm{aff}(\sigma) \to \mathrm{aff}(\tau)$ to be the affine map which agrees with F on σ. There is a matrix L_σ and a vector b_σ such that $F_\sigma(y) = L_\sigma y + b_\sigma$ for all $y \in \mathrm{aff}(\sigma)$. This *representation* is assumed to be with respect to the standard basis of R^n.

4.3.2 Definition

Let $H: \mathcal{G}^0 \to R^n$ be a map on the vertices of a triangulated n-manifold $(G,\mathcal{G})$. There is a unique extension of H to a piecewise affine map $\overline{H}: G \to R^n$ by taking $\overline{H}(x) = \sum_0^n \mu_i H(x_i)$, $\sum_0^n \mu_i = 1$, $\mu_i \geq 0$ where $x \in \sigma \in \mathcal{G}$, $x = \sum_0^n \mu_i x_i$ and $\sigma = \mathrm{conv}\{x_0, x_1, \ldots, x_n\}$. $\overline{H}$ is called a *piecewise affine extension* of H. When there is no ambiguity

we will denote the extension to C by the same symbol which denotes the function on G^0. Note that $(J,\mathcal{J}) = (R^n,[R^n])$ is assumed.

4.3.3 Lemma.

Given a piecewise affine map $H: G \to J$ where $(G,\mathcal{G})$ is a manifold in R^n let $\sigma \in \mathcal{G}$ have vertices $x_0, x_1, x_2, \ldots, x_n$, (thought of as columns) and let $W = (x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0)$ and $A = (H(x_1) - H(x_0), \ldots, H(x_n) - H(x_0))$ be $n \times n$ matrices. Then $L_\sigma = A\, W^{-1}$ and $b_\sigma = H(x_0) - A\, W^{-1} x_0$. See 4.3.1.

Proof. Let $x = x_0 + W\mu \in \sigma$ where $\mu = (\mu^1, \mu^2, \ldots, \mu^n)$. Hence $\mu = W^{-1}(x - x_0)$, $H_\sigma(x) = H(x_0) + A\mu = H(x_0) + A\, W^{-1}(x - x_0)$.

Another important idea we shall use is that of piecewise affine approximations of continuous maps with respect to a triangulation.

4.3.4 Definition:

Let $H: G \to J$ be continuous, a piecewise affine map $\hat{H}: G \to J$ which agrees with H on the vertices of $\mathcal{G}$ (a triangulation of G) is called a *piecewise affine approximation* to H with respect to the triangulation $\mathcal{G}$. Given such an H and $\hat{H}$, we have:

4.3.5 Lemma

If $\| x - x' \| \leq \delta$ implies $\| H(x) - H(x') \| \leq \varepsilon$ and if mesh of $\mathcal{G} \leq \delta$ then $\| H(x) - \hat{H}(x) \| \leq \varepsilon$ for all $x \in G$.

Proof. See Eaves [32, p. 90]

In following the paths of equilibria using complementary (simplicial) pivoting, piecewise affine approximations play an important role.

4.4 Eaves Algorithm with a Boundary Start

To provide a perspective for the algorithms of the next chapter the algorithm of Eaves [32] is given as it is implemented with a

boundary start This algorithm forms a basic component of our algorithms in their implementation.

The algorithm is based on 1-manifolds neat in $(G,\mathcal{G})$ which are inverse images of points $y \in J$ under a piecewise affine map $F : G \to J$, where G is an $(n+1)$-manifold and J and n-manifold. To ensure that $F^{-1}(y)$ is a 1-manifold the point y has to satisfy a weak *regularity* condition. Fortunately the same regularity condition ensures the neatness of these 1-manifolds in $(G,\mathcal{G})$.

4.4.1 Definitions:

Let G, J and F be as in the paragraph above. A point x in G is said to be a *degenerate* (otherwise *regular*) point of F if x lies in a cell $\sigma \in \mathcal{G}$ with $\dim F(\sigma) < n$. A value y in $F(G)$ is defined to be *degenerate* (otherwise *regular*) *value* if $F^{-1}(y)$ contains any degenerate points.

A set is regular if each of its points is regular. Clearly for a point x to be regular it is necessary that it lie interior to a cell of dimension n or $(n+1)$. The next theorem forms the main tool for the Eaves algorithm.

4.4.2 Theorem:

If y is a regular value, then $F^{-1}(y)$ is a 1-manifold neat in $(G,\mathcal{G})$, and $F^{-1}(y)$ is subdivided by sets of the form $\sigma \cap F^{-1}(y) \neq \phi$, $\sigma \in \mathcal{G}$.

Proof: See Eaves [32, p. 90]

Using this theorem and Lemma 4.2.7 we have the next lemma.

4.4.3 Lemma:

If y is regular and $\mathcal{G}$ finite, the number of boundary points plus the number of rays of $F^{-1}(y)$ is even.

Now under the conditions of this lemma, if $x_0 \in F^{-1}(y) \cap \mathrm{bd}\ G$ then the component of $F^{-1}(y)$ starting at x_0 will terminate in another boundary point or a ray. Further if rays are ruled out in $F^{-1}(y)$ by construction then the component will terminate in another point in bd G.

4.4.4 Remark

$(G,\mathcal{G})$ and $(J,\mathcal{J})$ are subdivided manifolds of dimension $(n+1)$ and n respectively. $F: G \to J$ is *piecewise affine* and y a regular value. Let $x_0 \in F^{-1}(y)$, σ_0 and $(n+1)$-cell of $\mathcal{G}$ such that $x_0 \in \sigma_0$, and a direction u_0 of $F^{-1}(y)$ in σ_0 the algorithm of [32] describes a strategy for moving along a curve of $F^{-1}(y)$ containing x_0 in the direction u_0. The algorithm termination depends on the nature of the curve of $F^{-1}(y)$ in the direction u^0 and G. The algorithm may terminate in a finite number of steps with one of the three possibilities, i.e., a 'Ray," 'Boundary" or 'Loop;" or the algorithm may *not* terminate in a finite number of steps in which case the sequence of points generated by the algorithm tends to infinity. These four (general) possibilities are sketched in Figure 1 on the next page.

4.4.5 Remark

Note that a "loop" can be ruled out if $x_0 \in \mathrm{bd}\ G$. If G is finite, then the algorithm does terminate in a finite number of steps. Then if $x_0 \in \mathrm{bd}\ G$ then finite termination in bd G or a 'ray' is guaranteed. If further a 'ray' termination can be ruled out (e.g., if G is bounded or if $F^{-1}(y)$ is bounded) then the algorithmic path has a simple structure of starting and terminating in bd G. This observation will play an important part in the algorithms of Chapter 5.

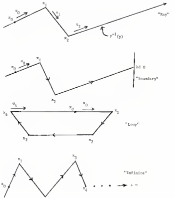

Figure 1 Termination Possibilities

Now we sketch briefly the algorithm of [32] for regular values as implemented with a boundary start.

4.4.6 Algorithm:

Let $x_0 \in F^{-1}(y) \cap \text{bd}\, G$ where y is a regular value, $x_0 \in \sigma_0 \in G$ and L_{σ_0} (as in 4.3.3) provide a direction $u_0 \neq 0$ such that $L_{\sigma_0} u_0 = 0$ and u_0 *points into* σ_0 (i.e., $x_0 + \theta u_0 \in \sigma_0$ for all sufficiently small $\theta > 0$). Thus we have the initial triple. Go to iteration $r = 0$.

Iteration $r = 0,1,2,\ldots$: From the triple (x_r, σ_r, u_r) compute $\delta^r = \sup\{\delta \mid x_r + \delta u_r \in \sigma_r\}$ and record $(x_r, \sigma_r, \delta^r)$. If $\delta^r = +\infty$, register a "*ray*" termination and stop. Otherwise set $x_{r+1} = x_r + \delta^r u_r$. If $x_{r+1} \in \text{bd}\, G$, register a *boundary* termination and stop. Otherwise determine $\sigma_{r+1} \in G \sim \{\sigma_r\}$ which contains x_{r+1}. Compute $u_{r+1} \neq 0$ where $L_{\sigma_{r+1}} u_{r+1} = 0$ and u_{r+1} points into σ_{r+1}. Go to iteration $(r+1)$ with the triple $(x_{r+1}, \sigma_{r+1}, u_{r+1})$.

4.4.7 Remark

Given a G and τ a facet of an $(n+1)$-cell $\sigma \in G$ one should have a systematic way of obtaining an $(n+1)$-cell $\rho \in G \sim \{\sigma\}$ which contains τ or determine that no such ρ exists. Such rules, called *pivoting rules*, are specified for various commonly used triangulations. Such a simplicial pivot step forms a component of each iteration r of the simplicial pivoting algorithms.

4.4.8 Remark

Details of implementing the algorithm will depend of course on the nature of G and F. In the algorithms of Chapter 5 steps similar to Iteration r of Algorithm 4.4.6 is executed by a mere (lexicographic) feasible pivot as in the Simplex method of Linear Programming [15].

The pivoting rules of familiar triangulations would serve well for the simplicial pivot step in our algorithm even though G is not always a simplicial subdivision.

4.4.9 Remark:

Note that Algorithm 4.4.6 was specified for regular values. An elaborate version is described in [32] for handling degenerate values. Conceptually the idea for treating a degenerate value y is to perturb it to a regular value $y + Y[\varepsilon]$ where $[\varepsilon]$ is a vector $(\varepsilon, \varepsilon^2, \varepsilon^2, \ldots, \varepsilon^n)$ of powers of ε and $Y = [y_1, y_2, \ldots, y_n]$ is a matrix of rank n. At the implementation level the method is to use the lexicographic technique for resolving degeneracies. For details, see [32, p. 111]. Similar ideas will be incorporated for dealing with degenerate values in the algorithms of the following chapter. Majthay [82, 84] and Eaves [28] also treat lexicographic systems in the context of complementary pivoting.

CHAPTER 5

THE ALGORITHMS

5.1 Overview

Three basic algorithms are devised in this chapter which can be used (depending on the model of the economy and the characterization of its equilibria) to trace an approximate equilibrium path when the economy is changed from its initial state at time t_0 to its final state at t_1 according to a specified dynamics. If the dynamics are not known or not specified then any homotopy (see 2.3) between the two economies provides possible deformations of the initial economy into the final one; in this case the algorithms trace possible equilibrium paths. The algorithms can also be adapted to the case when the initial economy and the dynamics of evolution is specified but the terminal point is left free. Thus the algorithmic output provides a path of equilibria on the semi-open interval $[t_0, t_1)$ where t_1 may be $+\infty$.

5.1.1 The general strategy

The underlying method for all the algorithms is the homotopy technique described in 4.1. The general strategy of the algorithms can be loosely described as follows. The economy will be specified as a function (or a point-to-set map) $f: M \to N$, where M is S in the first two algorithms and R_+^n in the third, N is S in the first algorithm, C(S) (as in 2.4.2) in the second algorithm and R^n in the third. The deformation of the economy can be specified as a function,

$F: M \times [t_0,t_1] \to N$. The function F is so constructed that for an appropriately chosen $y \in N$, $F^{-1}(y)$ contains the path of equilibria under study. To trace required paths in $F^{-1}(y)$ we use the method of complementary pivoting. So we construct H, a piecewise affine approximation of F with respect to a subdivision Π of $P = M \times [t_0,t_1]$. For the first two algorithms the subdivisions used are triangulations. The algorithm is initiated at x_0, the known equilibrium of the initial economy, in $M \times \{t_0\}$. x_0 is artifically made the *unique* point in $(M \times \{t_0\}) \cap H^{-1}(y)$ by the construction of H. Then using complementary pivoting (an algorithm similar to 4.4.6) we trace a path in $H^{-1}(y)$ starting at x_0. The path will terminate in $M \times \{t_1\}$ at an equilibrium of the final economy. This is guaranteed by remark 4.4.5 and the formulation of the problem such that

(a) $H^{-1}(y)$ contains no "rays," and

(b) $H^{-1}(y) \cap \{(\text{Bd } N) \times [t_0,t_1]\} = \phi$.

The above strategy underlies each one of the algorithms to be described. The construction of Π, H, etc. would be different in each of the algorithms and will depend on the description of the economy and the characterization of the equilibria.

5.1.2 Remark

The time intervals over which the change of economies are considered might be $[t_0,t_1]$ or $[t_0,t_1)$ where t_1 may be $+\infty$. In the description of the algorithms we shall always consider the intervals [0,1] or [0,1). This entails no loss of generality since by the use of a homeomorphism the functions involved can be reparametrized.

5.1.3 Remark

For each algorithm two types of questions will be discussed on the sense of the approximation obtained. The first, foremost in computational considerations, is the closeness of the approximation of the equilibria in relation to the mesh of the subdivision Π. The second, of conceptual importance, is whether the approximate equilibria converges to the actual equilibria in a limiting sense, i.e., when the mesh of Π is reduced to zero.

5.1.4 Triangulations

Triangulations (as in 4.2.2) form the basis of subdivisions Π; their pseudomanifold property underlies *complementary pivoting*.

For our algorithms triangulations of $S \times [0,1]$, $S \times [0,\infty)$ or $S \times (0,1]$ will play an important part. Various useful such triangulations have appeared in recent literature [30, 34, 47, 73, 91, 123-124, 131]. For a comprehensive discussion of triangulations, their properties, construction and appropriate rules for simplicial pivoting, see Todd [123, Chapters III and X]. Some general comments about the triangulations to be used in the subsequent sections are in order.

Let G be a triangulation of $S \times [0,1]$ or $S \times (0,1]$ under reference, and G^i be the i-faces of the triangulation. If $1 = \tau_0 > \tau_1 > \tau_2 > \tau_3 > \dots > 0$, then $S \times \{\tau_k\}$ for $k = 0, 1, 2, \dots$ is called the k-level, denoted by $S(k)$. All vertices of G^0 lie on $S(k)$ for some k. Let G_k, for $k = 0, 1, 2, \dots$ be defined as follows. $G_k = \{\sigma \in G^n : \sigma \subset S(k)\}$. Then G_k triangulates $S(k)$ for each $k = 0, 1, 2, 3, \dots$. Further, if $\sigma \in G$, then $\sigma \subset S \times [\tau_k, \tau_{k+1}]$ for some $k \geq 0$. Let δ_k be the mesh of G_k. Such triangulations have been discussed in detail in Eaves [30] and Todd [123-125].

In the algorithms two types of triangulations with differing behavior of δ_k can be used. If δ_k is uniform for $k = 0, 1, 2, \ldots$ then the triangulation will be called a *uniform triangulation*. If $\delta_k \to 0$ as $k \to \infty$ it will be called a *refining triangulation*. Merrill [91] Kuhn and MacKinnon [73], Kuhn [71], Gauld and Tolle [47] and Todd [125] contain versions of uniform triangulations. Eaves [30], Eaves and Saigal [34] and Todd [123-125] contain detailed discussions of refining triangulations. These references also contain specific rules for simplicial pivoting for the various triangulations.

5.2 Algorithm for the Exchange Economy

In this section we devise an algorithm to trace an equilibrium path of exchange economies (as in Section 3.2) under deformation. Recall that these economies could be specified in an aggregate form using the excess demand functions $g: S \to R^n$ and their equilibria could be characterized (by Definition 3.2.4) as $\bar{p} \in S$ such that $g(\bar{p}) \leq 0$.

Now assume we have $g(\cdot,t): S \to R^n$ which represents the economy at time t, for all $t \in [0,1]$, in other words, $g: S \times [0,1] \to R^n$ representing the change in economies from $g(\cdot,0)$ to $g(\cdot,1)$ is specified. $(\bar{p},t)$ would be an equilibrium at time t if $g(\bar{p},t) \leq 0$. We construct a map $f: S \times [0,1] \to S$ that the fixed points of $f(\cdot,t)$ coincide with the equilibria of the economy for all $t \in [0,1]$.

5.2.1 Construction

$f: S \times [0,1] \to S$ is defined as follows. For $(p,t) \in S \times [0,1]$ set for $i = 1, 2, \ldots, n$

$$f^i(p,t) = \frac{p^i + \max[0, g^i(p,t)]}{1 + \sum_{\ell} \max[0, g^{\ell}(p,t)]}$$

where f^i and g^i denote the i-th component of f and g respectively. Note that in this construction $\max[0,g^i(p,t)]$ etc., stand for the magnitude of the functions; they are dimensionless like the relative price components p^i.

A fixed point of $f(\cdot,t)$ for any $t \in [0,1]$ represents the equilibrium of the economy defined by excess demand functions $g(\cdot,t)$. To see this, for any t, at $\bar{p}$ such that $f(\bar{p},t) = \bar{p}$ we have, $c\bar{p}^i = \bar{p}^i + \max[0,g^i(\bar{p},t)]$ where $c = 1 + \sum_{\ell} \max[0,g^{\ell}(\bar{p},t)]$. If $c > 1$, then $\bar{p}^i(c-1) = \max[0,g^i(\bar{p},t)]$ implying that $g^i(\bar{p},t) > 0$ for all i with $\bar{p}^i > 0$. Since $\bar{p}^i > 0$ for some i, and $\bar{p} \geqq 0$, $\sum_i \bar{p}^i g^i(\bar{p},t) > 0$ which violates Walras law, i.e., $p^T g(p,t) = 0$ for all $p \in S$, $t \in [0,1]$.

Thus $c = 1$, and $g(\bar{p},t) \leqq 0$.

This construction is based on Scarf [107, p. 30]. There are other possible constructions, e.g., Arrow and Hahn [6, p. 27] which lead to a suitable f.

5.2.2 <u>Remark</u>:

Thus, by construction above, the problem we have reduce to computing a path of ε-approximation fixed points of f. More precisely, given an $\varepsilon > 0$, the algorithm will compute a continuous path, $\phi = (\phi_1,\phi_2)$: $[0,1] \to S \times [0,1]$ such that $\phi_2(0) = 0$, $\phi_2(1) = 1$ and $\| f(\phi(t)) - \phi_1(t)\| \leq \varepsilon$ for any $t \in [0,1]$. The strategy in the algorithm is to compute a path of the fixed points of $\bar{g}$: $S \times [0,1] \to S$ which is the piecewise affine approximation of f with respect to a triangulation G of $S \times [0,1]$ of mesh δ, where δ is such that $\| f(x,t) - f(x',t)\| \leqq \varepsilon$ whenever $\| x-x'\| \leqq \delta$.

5.2.3 <u>Lemma</u>

Fixed points of $\bar{g}(\cdot,t)$ provide ε-approximate fixed points of $f(\cdot,t)$ for all $t \in [0,1]$.

Proof: If $\alpha(t)$ is a fixed point of $\tilde{g}(\cdot,t)$ we have $\tilde{g}(\alpha(t),t)=\alpha(t)$. Since $\tilde{g}$ is a piecewise affine approximation of f, from Lemma 4.3.5, we have $\| f(\alpha(t),t) - \tilde{g}(\alpha(t),t)\| \leq \varepsilon$. Thus $\| f(\alpha(t),t) - \alpha(t)\| \leq \varepsilon$ as required.

Now we are ready to set up the algorithm. As a subdivision of $S \times \{0,1\}$ we use a uniform triangulation G of mesh δ. δ is chosen based on the given $\varepsilon > 0$ as in Remark 5.2.2.

5.2.4 Construction of H.

We construct the piecewise affine homotopy H as follows. Define $\bar{H}: G^0 \to S - S$, by

$$\bar{H}(p,t) = \begin{cases} f(p,t) - p, & \text{if } t > 0 \\ (p_0 - p) & \text{if } t = 0. \end{cases}$$

where (p,t) is a vertex of G^0 in $S \times \{0,1\}$, $(p_0,0)$ is the equilibrium point of the initial economy. (See Remark 5.2.13). Notice that H on $S \times \{0\}$ is artificially specified such that $(p_0,0)$ is the unique solution to $H(p_0,0) = 0$. If p_0 is, in fact, the unique equilibrium of the initial economy, i.e., the unique fixed point of $f(\cdot,0)$ then an artificial H on $S \times \{0\}$ is not necessary. Now extend $\bar{H}$ to the piecewise affine map $H: S \times \{0,1\} \to S - S$ as in 4.3.2.

We initiate the algorithm at the unique solution $(p_0,0)$ to $H(p,t) = 0$ on $S \times \{0\}$. Then a route is traced in $H^{-1}(0)$ until it terminates in $S \times \{1\}$ at a point of the form $(p_1,1)$. Such a route from $(p_0,0)$ to $(p_1,1)$ is the path of fixed points of $\tilde{g}$, and thus a path of ε-approximate fixed points of f. Thus we obtain the required path of approximate equilibria. The details of constructing this route are given in 5.2.8.

5.2.5 Remark

To guarantee that a route in $H^{-1}(0)$ initiated at a point $(p_0,0)=q_0$ in $S \times \{0\}$ would terminate in $S \times \{1\}$ we need two conditions (a) it does not have a "ray" termination, (b) it does not terminate in $(\mathrm{bd}\ S) \times [0,1]$. "Ray" termination is ruled out as $S \times [0,1]$ is compact. To ensure (b) we make the following assumption.

5.2.6 Assumption

For convenience we can assume that $f(\mathrm{bd}\ S) \subset \mathrm{int}\ S$ for $f(\cdot,t)$, all t, otherwise we can imbed S in a larger simplex S' and artifically obtain the condition. The impossibility of boundary equilibria has also been evoked based on economic arguments as, for example, in [21-23, 95]. An assumption of this sort will be made in all the three algorithms.

5.2.7 Remark

Provisions for handling degenerate values are incorporated in our algorithms. The underlying concept is to perturb the starting point q_0 (which is made a regular point by construction of H) to $\bar{q}_0$ such that $H(\bar{q}_0)$ is a regular value. In implementation, however, it is merely the use of lexicographic rules for the pivot steps in the appropriate (lexicographic) linear systems. In details, we follow Eaves [32]

5.2.8 Algorithm

Start: Given a triangulation G of $S \times [0,1]$ construct H as in 5.2.4. $(p_0,0)$ the given equilibrium of the initial economy is the unique element of $H^{-1}(0) \cap S \times \{0\}$ by construction. Given $(p_0,0) = q_0$, select the unique n-cell γ_0 in $S \times \{0\}$ of G which contains q_0 and let σ_0 be the unique (n+1)-cell containing γ_0. To ensure the regularity condition, if necessary, perturb q_0 to $q_0 + Q_0[\varepsilon]$ where Q_0 is a n × n

matrix of rank n and $[\varepsilon] = (\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n)$ so that the perturbed value is in γ_0 for all small $\varepsilon > 0$. Let $0 \neq u_0 \in \operatorname{Ker} L_{\sigma_0}$ such that u_0 points into σ_0 from $q_0 + Q_0[\varepsilon]$. $(q_0, Q_0, \sigma_0, u_0)$ is the initial quadruplet. Note that $[\varepsilon]$ plays no real part in the implementation, pivoting in the corresponding lexicographic system is what is involved. $H(q_0 + Q_0[\varepsilon])$ can be shown to be a regular value for all small $\varepsilon > 0$. With $(q_0, Q_0, \sigma_0, u_0)$ as input, go to iteration 0.

Iteration $r = 0, 1, 2, \ldots$. Given $(q_r, Q_r, \sigma_r, u_r)$ compute $b_r + B_r[\varepsilon] = \sup\{b : q_r + Q_r[\varepsilon] + b\, u_r \in \sigma_r\}$ for all small $\varepsilon > 0$. The fact that the chord in σ_r can be expressed as $b_r + B_r[\varepsilon]$ is clear in the context of perturbations and the corresponding lexicographic system (see [15] or [32, p. 110]). Let

$$q_{r+1} + Q_{r+1}[\varepsilon] = q_r + Q_r[\varepsilon] + (b_r + B_r[\varepsilon])u_r$$

for all small $\varepsilon > 0$. If this point is in $S \times \{1\}$ for small $\varepsilon > 0$, stop. Otherwise determine the unique $\sigma_{r+1} \in G \sim \{\sigma_r\}$ which contains $q_{r+1} + Q_{r+1}[\varepsilon]$ for small enough $\varepsilon > 0$. Determine $0 \neq u_{r+1} \in \operatorname{Ker} L_{\sigma_{r+1}}$ which points into σ_{r+1}. Proceed to iteration (r+1) with the quadruplet $(q_{r+1}, Q_{r+1}, \sigma_{r+1}, u_{r+1})$.

5.2.9 Remark:

Note that the possibility of a 'ray' termination or a 'boundary' termination in (bd S) × [0,1] is not even considered in the algorithm. The boundedness of S × [0,1] and Assumption 5.2.6 make it unnecessary to do so. Thus a termination in S × {1} is guaranteed in a finite number of iterations when G is a uniform triangulation.

This concludes the discussion of the basic algorithm. Now some of its properties and adaptations for solving related problems are discussed in the following remarks.

5.2.10 Remark:

If the economies are specified only at the initial and final levels, and the deformation of one to the other is unknown or unspecified we can use this algorithm to compute possible equilibrium paths connecting the equilibria of the initial economy to those of the former. Assume that the initial and the final economies are characterized, as in 5.2.1, by $f(\cdot,0)$ and $f(\cdot,1)$: $S \to S$. Now using Theorem 2.3.2 we can construct a homotopy $F: S \times [0,1] \to S$ such that $F(\cdot,0) = f(\cdot,0)$ and $F(\cdot,1) = f(\cdot,1)$. Since any such continuous function F would do as a possible homotopy any information about the behavior of the economy for t, $0 < t < 1$, can be incorporated in formulating F. A convenient, though perhaps rigid, homotopy always available is $F(p,t) = (1-t)f(p,0) + t\ f(p,1)$.

The algorithm, implemented using some such F, produces a possible path (of equilibria) of smooth transition from an equilibrium of the initial economy to one of the final economy.

5.2.11 Remark:

If the initial economy is specified along with $f: S \times [0,\infty) \to S$ the rule for evolution of the economy, by implementing the algorithm on a refining triangulation of $S \times [0,\infty)$ we could generate an equilibrium path $\phi: [0,\infty) \to S \times [0,\infty)$ of the evolving economy. The mesh of the triangulation of $S \times \{0\}$ is taken to be δ chosen as in 5.2.2. Eaves' triangulation K_1 and K_2 of [30] could form the basis of the required triangulation.

A more convenient implementation might be on a refining triangulation of $S \times [0,1)$ of mesh δ (as in 5.2.2). This would involve a straight-forward reparametrization $\tau: [0,\infty) \to [0,1)$

5.2.12 Algorithm

The algorithm can be initiated at $(p(0), \tau(0)) = (p_0, 1)$ in $H^{-1}(0)$. The algorithm generates a route which we can parametrize by $(p(t), \tau(t))$ where $t \in [0, +\infty)$ and $(p(\ell), \tau(\ell)) = q_\ell \in \sigma_\ell$ for $\ell = 0, 1, 2, \ldots$. Since the σ_ℓ's do not repeat and there are finitely many cells in $S \times [1, \varepsilon]$ for any $\varepsilon > 0$ we see that $\tau(t) \to 0$ as $t \to +\infty$. The rest of the details of implementation are as in 5.2.8. Note that even though the algorithm generates an infinite sequence of simplices, it can be terminated at any level t_1 with an equilibrium path $\phi: [0, t_1] \to S \times [0, t_1]$.

5.2.13 Remark

This algorithm can be used to compute an unknown equilibrium point of a given static economy $f: S \to S$, as is well known. Using a refining triangulation of $S \times (0,1]$ and the homotopy $f(p,t) = t(p - p_0) + (1 - t) f(p)$ where $(p_0, 1)$ is an arbitrarily chosen starting point, Algorithm 5.2.12 can be used to generate a path starting at $(p_0, 1)$ to an equilibrium point of the given economy as $t \to 0$. This method can be used to compute an equilibrium point of the initial economy in our basic algorithm, if it is unknown. Such an implementation of the homotopy technique clearly yields the homotopy principle.

Note that this remark contains a constructive proof of Theorem 2.4.1.

5.2.14 Remark

The homotopy path generated by the algorithm 5.2.8 cannot return to $S \times \{0\}$ as $(p_0, 0)$ is a unique solution to $H(p, 0) = 0$ by construction. For the remainder of the path there is the possibility of retrogression

if the economies at the various levels k have multiple approximate equilibria, i.e., $H(p,k) = 0$ has more than one solution. At a price however, this retrogression can be prevented. The strategy is as follows. At any level k, $k \neq 0$ let σ_k be the n-simplex containing the point p_k such that $H(p_k,k) = 0$. After σ_k has been encountered for the first time, use at the k-level, an artificial piecewise affine map such that (p_k,k) is the unique solution in $S \times \{k\}$ (instead of the originally defined H). $A(p-p_k)$ provides a convenient such map, where A is a suitable nonsingular matrix. The algorithmic path never returns to the k-level.

5.2.15 Remark

In the algorithm of 5.2.8, by letting $\varepsilon \to 0$, (ε as in Remark 5.2.2) we obtain a connected set meeting $S \times \{0\}$ and $S \times \{1\}$, providing a constructive proof of Browder's Theorem 2.4.3.

In the next section we devise an algorithm for economies characterized by point-to-set maps. The underlying strategy and many of the arguments of this section will apply to the algorithm of the next.

5.3 Algorithm for Point-to-Set Economies

In this section we consider the more general models of the economy discussed in 3.3. In these models the functions characterizing the economies are point-to-set maps, their equilibria (as discussed in 3.3) can be taken to be characterized by the Kakutani fixed points of an appropriate U.S.C. point-to-set map. So in this section, for convenience, we will characterize the economy at time t, as $f(\cdot,t): S \to C(S)$ where $C(S)$ is the set of the nonempty compact convex subsets of S, and f is an U.S.C. point-to-set map such that its fixed

points $\{x \in S: x \in f(x,t)\}$ coincide with the equilibria of the economy at time t. In these models S, the standard simplex in an appropriate Euclidean space, need not be the set of price vectors, see Remark 3.3.i. To reflect this difference from the earlier section, we will denote the generic point of S by x instead of p.

As before, $f: S \times [0,1] \to C(S)$ represents the change in the economy from the initial one $f(\cdot,0)$ to the final one $f(\cdot,1)$. Our algorithm would trace an approximate equilibrium path starting at an equilibrium $(x_0,0)$ of the initial economy and terminating at an equilibrium $(x_1,1)$ of the final economy. The underlying strategy is to compute a route of approximate fixed points of f. More precisely

5.3.1 <u>Problem</u>:

Given $f: S \times [0,1] \to C(S)$ such that $f(\cdot,t)$ characterizes the economy at time t, $t \in [0,1]$. The problem is to compute a path, $\phi = (\phi_1,\phi_2): [0,1] \to S \times [0,1]$ such that $\phi_2(0) = 0$, $\phi_2(1) = 1$ and given any $\varepsilon > 0$, $\phi_1(t) \in B(f(\phi(t)),\varepsilon)$ for all $t \in [0,1]$.

Before the details of the algorithm are discussed we need the ideas of ε-approximate fixed points and piecewise affine approximations of a point-to-set map $f: S \to P(S)$.

5.3.2 <u>Definition</u>

Let $f: S \to P(S)$ be a point-to-set map. Given $\varepsilon > 0$, $x \in S$ such that $x \in B(f(x),\varepsilon)$ is called an ε-approximate fixed point of f.

5.3.3 <u>Definition</u>:

Let f be as in 5.3.2 and G a triangulation of S, $f(x) \neq \phi$ for all $x \in S$. For each vertex y of G, pick $g(y) \in f(y)$ and let $f: S \to S$ be a piecewise affine map which assumes the preselected value $g(y)$ for all $y \in G^0$. This function $g: S \to S$ is well defined (see 4.3.2) and is

called a *piecewise affine approximation* to f with respect to G. Using local finiteness of G it is easy to show that g is continuous.

5.3.4 Lemma:

Let $f: S \to C(S)$ be U.S.C. For any $\varepsilon > 0$ choose $\delta > 0$ according to Definition 2.2.1 (ii). If g is a piecewise affine approximation of f with respect to any triangulation G of mesh δ then fixed points of g are ε-approximate fixed points of f.

Proof: From the choice of δ as in 2.2.1 (ii) we have for any $y \in B(x,\delta) \cap S$, $f(y) \subset B(f(x),\varepsilon)$, as f is U.S.C.

Now if $\bar{x}$ is a fixed point of g, then $\bar{x} = \sum_{i=1}^{n} \lambda_i y_i = \sum_{i=1}^{n} \lambda_i f(y_i)$, where $\lambda_i \geq 0$, and $\Sigma \lambda_i = 1$, and $\sigma = \text{conv}\{y_1, y_2, \ldots, y_n\} \in G$. Since the mesh of G is δ, $\| y_i - \bar{x} \| \leq \delta$ for all $i = 1, 2, \ldots, n$ $g(y_i) \in f(y_i) \subset B(f(\bar{x}),\varepsilon)$ for $i = 1, 2, \ldots, n$. Since $B(f(\bar{x}),\varepsilon)$ is convex it contains $\bar{x} = \Sigma \lambda_i y_i$, $\Sigma \lambda_i = 1$, $\lambda_i \geq 0$, i.e., $\bar{x} \in B(f(\bar{x}),\varepsilon)$ is an ε-approximate fixed point of f.

The above lemma indicates the approximation obtained for a particular run of the algorithm with δ as the mesh of G.

Now we are ready to set up the algorithm. Given $f: S \times [0,1] \to C(S)$ our algorithm will compute a path of ε-approximate fixed points of $f(\cdot,t)$ for all $t \in [0,1]$ and thus generate an approximate equilibrium path. The overall strategy is the same as in 5.2. We construct a piecewise affine homotopy $H: S \times [0,1] \to S - S$, such that $H^{-1}(0)$ contains a route of ε-approximate fixed points of f. We initiate the route at an equilibrium point $(x_0,0)$ of the initial economy in $H^{-1}(0)$ and trace the path in $H^{-1}(0)$ which terminates in $S \times \{1\}$. To ensure that the path will not terminate in $(\text{bd } S) \times [0,1]$ we make an assumption similar to 5.2.6.

5.3.5 Assumption:

For convenience we make the assumption that for all $x \in \mathrm{bd}\, S$, all $t \in [0,1]$, $f(x,t) \cap \mathrm{int}\, S \neq \phi$. This involves no loss of generality, since by imbedding S in a larger simplex such a condition could always be artificially obtained, as for example in [27].

5.3.6 Construction

Given G as in 5.3.4, we construct a piecewise affine map H such that its zeros coincide with the fixed points of piecewise affine approximations of f. Define $H: S \times [0,1] \to S - S$ as follows. For $(x,t) \in S \times [0,1] \cap G^0$

$$H(x,t) = \begin{cases} g(x,t) - x & \text{for } t > 0, \text{ where } g(x,t) \in f(x,t) \cap \mathrm{int}\, S \\ (x - x_0) & \text{for } t = 0. \end{cases}$$

Note that $g(x,t)$ is an arbitrary vector in $f(x,t) \cap \mathrm{int}\, S$ which is nonempty by 5.3.5.

Now the required homotopy $H: S \times [0,1] \to S - S$ is obtained by a piecewise affine extension of H (on G^0) above, using 4.3.2.

5.3.7 Algorithm:

Given f, H and G as above, the algorithm is initiated at $(x_0, 0)$ the unique zero of H in $S \times \{0\}$. Let $\tau_0 \subset S \times \{0\}$ be the unique n-simplex of G which contains x_0 (or its perturbed value, $x_0 + X(\varepsilon)$ as in 5.2.8) in its interior, and σ_0 the (n+1)-simplex containing τ_0. The algorithmic route, constructed by a procedure identical to that in 5.2.8, terminates at some point of $S \times \{1\}$. The compactness of $S \times [0,1]$ precludes a "ray" termination, Assumption 5.3.5 and construction of H implies that $H^{-1}(0) \cap [(\mathrm{bd}\, S) \times [0,1]] = \phi$, thus preventing a boundary termination in $(\mathrm{bd}\, S) \times [0,1]$. Thus the algorithm terminates in $S \times \{1\}$ in a finite number of steps when G is finite.

5.3.8 Remark

When the dynamics of deformation is not specified, and only the initial and final economies are given, a possible dynamics can always be constructed artificially using homotopy theory. By using Remark 5.2.10 possible equilibrium paths of transition from the initial economy to the final one can be constructed. For details, see 5.2.10.

5.3.9 Remark

When $f: S \times [0,\infty) \to C(S)$ defines the evolution of the economy an approximate equilibrium path of evolution $\phi: [0,\infty) \mapsto S \times [0,\infty)$ can be generated (up to any desired value of t) using the algorithm on a refining triangulation of $S \times [0,\infty)$ or $S \times (0,1]$. For details, see Remarks 5.2.11 and 5.2.12.

5.3.10 Remark

A special case of the algorithm can be used to compute the unknown equilibrium point of a given static economy. The details are similar to that in Remark 5.2.13. Note that this special case reduces to the algorithm using homotopy principle of Eaves [30, 32], Eaves and Saigal [34], Merrill [91], etc. Thus our algorithm also provides a constructive proof of Theorem 2.4.2.

5.3.11 Remark

The retrogression of the algorithmic path can be prevented, if necessary, by sequential reconstruction of H as detailed in Remark 5.2.14.

5.3.12 Remark

By letting $\varepsilon \to 0$ (where ε is as in Lemma 5.3.4) in the algorithm 5.3.7 we obtain the set guaranteed in Theorem 2.4.4. This follows from $f(\cdot,t)$ being U.S.C. for each $t \in [0,1]$ and $f(x,t)$ being nonempty

compact convex sets for each $(x,t) \in S \times [0,1]$. Thus the algorithm provides a constructive proof of Theorem 2.4.4.

5.4 Algorithm for the Walrasian Model with Production

In this section an algorithm is described which traces the equilibrium path of a Walrasian competitive general equilibrium model with production (discussed in 3.4) under deformation. The underlying strategy of the algorithm is the use of homotopy technique as in the earlier algorithms (of 3.2 and 3.3) but the details of implementation are quite different and more involved. One way of viewing the algorithm is as a *parametric* version of Scarf's algorithm [107] when it is implemented with a simplicial subdivision of S instead of the original framework of *primitive sets*. In this sense the applications of the algorithm of this section are quite wide-ranging and include *parametric* versions of various problems to which Scarf's Theorem 4.2.3 and his algorithm [107] has been addressed. But in the following discussion we develop the algorithm in the specific context of parametrically changing Walrasian economies.

Recall that equilibria of a Walrasian economy (d,w,A) are characterized by Definition 3.4.2, as $(\overline{p},\overline{y})$ such that

(a) $d(\overline{p}) = A\overline{y} + w$

(b) $\overline{p}^T A \leq 0$, where $\overline{p}$ is the vector of equilibrium prices $\overline{y}$ the vector of equilibrium activity levels, d the aggregate demand function, w the total resource endowment of the economy and A the activity analysis matrix, all as discussed in 3.4.

5.4.1 Problem

The problem is to devise an algorithm to generate an approximate equilibrium path $(\overline{p},\overline{y}): [0,1] \rightarrow S \times R^k_+ \times [0,1]$ such that $(\overline{p}(t),\overline{y}(t))$

is the equilibrium configuration of the economy $(\theta(\cdot,t), w(t), A(t))$ at time t, for all $t \in [0,1]$

As in the earlier sections 5.2 and 5.3, a piecewise affine homotopy H will be defined on an appropriate subdivided manifold. But the construction of the manifold and the homotopy H are quite different from that of the earlier algorithms, and in comparison, more involved.

5.4.2 Construction of the subdivided manifold

The subdivided manifold we use is $(R^n_+ \times [0,1], \overline{G})$, where $\overline{G}$, a subdivision of $R^n_+ \times [0,1]$, is constructed as follows. First a triangulation G of $S \times [0,1]$ is constructed. For any simplex $\sigma \in G$, where $\sigma = \text{conv}\{y_1, y_2, \ldots, y_m\}$ we construct $\overline{\sigma} \in \overline{G}$, the cell generated by σ as follows. For each vertex $y_i = (x_i, t_i)$ the ray $r(y_i) = \{(\lambda x_i, t_i), \lambda \geq 0\}$ is generated, then $\overline{\sigma} = \text{conv}\{r(y_1), r(y_2), \ldots, r(y_m)\}$. The mesh size, δ is of course, always in reference to that of G.

It is easy to see that $(R^n_+ \times [0,1], \overline{G})$ is a subdivided manifold.

5.4.3 Construction of H

We start with $H: G^0 \to R^n$ (defined on the vertices of G). First the piecewise affine extension of H to $S \times [0,1]$ is constructed as in 4.3.2. Then for each point $(x,t) \in S \times [0,1]$, set $H(\lambda x, t) = \lambda H(x,t)$ all $\lambda \geq 0$, thus extending H to $R^n_+ \times [0,1]$.

5.4.4 Lemma

H is piecewise affine with respect to $\overline{G}$.

Proof. For $u, v \in \overline{\sigma} \in \overline{G}$, and $0 \leq \delta \leq 1$ we should obtain,

$H(\delta u + (1-\delta)v) = \delta H(u) + (1-\delta)H(v)$.

Let $c = \delta a + (1-\delta)b$ where $u = ax$, $v = by$, $a, b \geq 0$; $x, y \in \sigma \in G$

Then,

$$\begin{aligned} H(\delta u + (1-\delta)v) &= H(c(\frac{\delta a}{c}\, x + \frac{(1-\delta)b}{c}\, y)) \\ &= cH(\frac{\delta a}{c}.x + \frac{(1-\delta)b}{c}\, y) \\ &= \frac{c\delta a}{c}\, H(x) + \frac{c(1-\delta)b}{c}\, H(y) \\ &= \delta a\, H(x) + (1-\delta)b\, H(y) \\ &= \delta H(u) + (1-\delta)Hv. \end{aligned}$$

5.4.4 Remark

From construction 5.4.3 it is clear that specifying H on G^0 uniquely defines a piecewise affine map H on $R^n_+ \times [0,1]$ (which is piecewise affine with respect to the subdivision $\overline{G}$). In many applications of the alagrithm the specification of H on G^0 will be characterized by the nature of the application.

Now using the above constructions we are ready to describe the algorithm of this section. Given a piecewise affine homotopy $H: R^n_+ \times [0,1] \to R^n$ and a regular value $\bar{w}$ in R^n, the required route will be constructed in $H^{-1}(\bar{w})$ starting at α_0 in $R^n_+ \times \{0\}$ and terminating at a point α_1 in $R^n_+ \times \{1\}$. From such a path $\alpha: [0,1] \to R^n_+ \times [0,1]$ we will construct the required equilibrium path $(\bar{p},\bar{y}): [0,1] \to S \times R^k_+ \times [0,1]$ using 5.4.8.

As usual an assumption ruling out boundary equilibria will be made. Lemma 5.4.12 will ensure that $H^{-1}(\bar{w})$ contains no rays. Together they guarantee the termination of the algorithmic path in $R^n_+ \times [1]$.

Now we have all the tools required to set up the algorithm of this section to trace the approximate equilibrium path of a changing economy specified as $(\vartheta(\cdot,t), w(t), A(t))$ at time t, for all $t \in [0,1]$.

We make the following assumption.

5.4.5 Assumption

A boundary assumption is made similar to 5.2.6 and 5.3.5 without involving a loss of generality. We assume $Hx = w$ has no solution in $(\mathrm{bd}\ R^n_+) \times [0,1]$, otherwise we can imbed the problem and obtain the condition artifically by specifying H on $(\mathrm{bd}\ R^n_+) \times [0,1]$ suitably.

5.4.6 Assumption

Assumption 3.4.1 we made for the Walrasian model is assumed to hold for all $t \in [0,1]$, i.e., $\{y \mid A(t)y + w(t) \geq 0,\ y \geq 0\}$ is bounded for all $t \in [0,1]$. This mild assumption is the usual one stipulating that at all times the set of the activity levels that give rise to a nonnegative net supply of all commodities is bounded.

5.4.7 Assumption:

Here we assume that $w(t) = w(0) = w > 0$, for all $t \in [0,1]$. This assumption is made merely for *convenience* in describing the basic algorithm. This assumption will be dropped altogether and the general case treated in 5.4.32.

Let us denote by $\alpha: [0,1] \to R^n_+ \times [0,1]$ the path generated by the algorithm. For any $t \in [0,1]$, by a suitable interpretation of $\alpha(t)$, we will extract the required equilibrium configuration at t, $(\overline{p}(t), \overline{y}(t))$, see 5.4.9.

5.4.8 Construction of the homotopy

$(R^n_+ \times [0,1], \overline{G})$ is a subdivided manifold constructed as in 5.4.2. G is a triangulation of $S \times [0,1]$ which generated $\overline{G}$. For any $(p,t) \in G^0$, $H(p,t)$ is prescribed as follows.

Set $\pi(p,t) = \max\limits_{1 \leq j \leq k} p^T A_j(t)$, i.e., the maximum profit from all the activities $j = 1, 2, \ldots, k$. Choose some $\ell = \ell(p,t)$ such that $p^T A_\ell(t) = \pi(p,t)$.

Then define $H(p,t)$ as,

$$H(p,t) = \begin{cases} -A_{\ell}(t) & \text{if } \pi(p,t) > 0, \\ d(p,t) & \text{if } \pi(p,t) \leqq 0 \end{cases}$$

Extend H to $R_+^n \times [0,1]$ using 5.4.3.

Now let $\tilde{w}$ be a regular value of H obtained (if necessary) by perturbing w in 5.4.7. For simplicity of presentation we will assume $\tilde{w} = w$ in the following discussion. Starting from α_0 in $H^{-1}(\tilde{w})$ we trace a path in $H^{-1}(\tilde{w})$. The details of construction of such a path is given in 5.4.25.

The following remarks give a motivation for the constructs used and a way of extracting $(\bar{p}(t), \bar{y}(t))$ from $\alpha(t)$ for each $t \in [0,1]$.

5.4.9 <u>Remarks</u>

Since $\tilde{w}$ is a regular value of H the points of $H^{-1}(\tilde{w})$ lie interior to $(n+1)$-cells or n-cells of $\bar{G}$. Now consider a $t \in [0,1]$ such that $\alpha(t)$ lies in an n-cell $\bar{T} \in \bar{G}$, and $\bar{T} \in \bar{G}$ be the simplex generating $\bar{T}$. Let $T = \text{conv}\{u_1, u_2, \ldots, u_n\}$, $u_i \in S \times [0,1]$ of the form (p_i,t).

From construction 5.4.3 it is clear that $\alpha(t)$ can be uniquely expressed as a positive combination of the u_i's, as $\alpha(t) = \sum_i \lambda^i(t)u_i$, $\lambda^i > 0$, $i = 1, 2, \ldots, n$ since H is linear on $\bar{T}$ we have

(5.4.10) $\quad H(\alpha(t)) = \sum_i \lambda^i(t)\, H(u_i) = w$

Now that the algorithm in its implementation uses linear systems of the form given above. Therefore $\lambda^i(t)$ are the data generated and handled by the algorithm even though $\alpha(t)$ is used for its presentation.

Now if $\bar{T}$ is in $R_+^n \times \{t\}$ for some $t \in [0,1]$ then depending on the definition of H in 5.4.8, $H(u_i)$ in the equation (5.4.10) can be grouped into two sets indexed by Δ and Γ as follows.

$H(u_i) = H(p_i,t) = d(p_i,t)$ for $i \in \Delta = \{1, 2, \ldots, n\}$

$H(u_i) = H(p_i,t) = -A_{\ell_i}(t)$ for $i \in \Gamma \subset \{1, 2, \ldots, n\}$

$\ell_i = \ell(p_i,t)$ as in 5.4.8.

Using the above grouping, (5.4.10) can be written as,

$$(5.4.11)\quad -\sum_{\Gamma} \lambda^i(t)\, A_{\ell_i}(t) + \sum_{\Delta} \lambda^i(t)\, d(p_i,t) = w$$

$$\lambda^i(t) > 0, \text{ all } i.$$

Now it will be shown that for each t, $\lambda^i(t)$ $(i \in \Gamma)$ approximates the activity levels of the activities ℓ_i used, further that $\sum_{\Delta} \lambda^i$ is approximately equal to 1, and that a vector $\overline{p}(t) \in \tau$ approximates the equilibrium vector of prices.

But before we show that, we demonstrate that systems of the form (5.4.11) are bounded.

5.4.12 Lemma:

Let $0 < \varepsilon \leq \frac{1}{2}(\min_{1 \leq i \leq n} w^i)$ and δ be such that $|d(p,t) - d(p',t)| \leq \varepsilon$ for all $p, p' \in S$, $t \in [0,1]$. For any triangulation δ of mesh G or less, for any system of the form (5.4.11) the set $\Lambda = \{\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^n) \mid \lambda \text{ solves } (5.4.11)\}$ is compact.

Proof: Let ε and δ be as given. Recall τ is an n-simplex in $S \times \{t\}$ such that $\tau = \mathrm{conv}\{(p_1,t), (p_2,t), \ldots, (p_n,t)\} \in G$. Clearly for any $j, k \in \{1, 2, \ldots, n\}$ we have $|p_j - p_k| \leq \delta$ and $|d(p_j,t) - d(p_k,t)| \leq \varepsilon$.

Then,

$$(5.4.13)\quad |p_j^T d(p_j,t) - p_j^T d(p_k,t)| \leq |p_j|\,|d(p_j,t) - d(p_k,t)|$$

$$\leq |d(p_j,t) - d(p_k,t)| \leq \varepsilon.$$

Now, $p_j^T d(p_j,t) = p_j^T w$ (by Walras law)

$$\geq \min_{1 \leq i \leq n} w^i > \varepsilon.$$

Thus from (5.4.13) we have,

$$(5.4.14) \quad p_j^T\, d(p_k,t) > 0 \text{ for all } k \in \{1, 2, \ldots, n\}$$

Now consider system (5.4.11), by the construction of H (see 5.4.8) we have for any $j \in \Gamma$, $p_j^T A \leq 0$. Multiply (5.4.11) by p_j for some $j \in \Gamma$ and rearrange to obtain,

$$(5.4.15) \quad \sum_{\Delta} \lambda^i(t)\, p_j^T\, d(p_i,t) = p_j^T w + \sum_{\Gamma} \lambda^i(t)\, p_j^T A_{\ell_i}(t)$$

$$\sum_{\Delta} \lambda^i(t)\, p_j^T\, d(p_i,t) \leq p_j^T w$$

Using (5.4.14) we have

$$(5.4.16) \quad 0 \leq \lambda^i(t) \leq \frac{p_j^T w}{p_j^T\, d(p_i,t)} \text{ for all } i \in \Delta$$

Thus $\lambda^i(t)$ is bounded, for $i \in \Delta$. The boundedness of Λ, then, follows using assumption 5.4.6. Since Λ is also closed, it is compact.

5.4.17 <u>Remark</u>

In this remark we pursue a limiting argument which asserts that as the mesh δ is reduced to zero the relevant parameters of 5.4.11 converges to the required equilibrium values at time t.

Consider the sequence of triangulations G^k of mesh δ^k, $k \in K$ such that $\delta^k \to 0$. Let τ^k be the n-simplex in G^k as in remark 5.4.9. Using the compactness of S, d(S,t) and Λ we can evoke the standard arguments to show that there is a subsequence $L \subset K$ over which $\tau^k \to \bar{p}(t) = \bar{p}$, $\lambda_k^i(t) \to \lambda^i(t)$ and $d(p^k,t) \to d(\bar{p},t)$, $p^k \in \tau^k$. In the limit (5.4.11) can be written as,

$$(5.4.18) \quad -\sum_{\Gamma} \lambda^i(t)\, A_{\ell_i}(t) + \sum_{\Delta} \lambda^i(t)\, d(\bar{p},t) = w$$

$$\lambda^i(t) > 0 \quad i = 1, \ldots, n.$$

Set $\zeta(t) = \sum_{\Delta} \lambda^i(t)$, define $y(t) \in R^k$, the vector of activity levels, as follows

$$y^j(t) = \begin{cases} \lambda^i(t) & \text{if } j = \ell_i \text{ for some } i \in I \\ 0 & \text{otherwise} \end{cases}$$

Using the new variables, (5.4.18) becomes,

(5.4.19) $\quad -A(t)\,\overline{y}(t) + \zeta(t)\, d(\overline{p},t) = w$

$$\overline{y}(t) \geqq 0,\ \zeta(t) \geqq 0$$

From the definition of H, we have

(5.4.20) $\quad$ if $\zeta(t) > 0 \quad \overline{p}^T A(t) \leqq 0$, and

if $\overline{y}^j(t) > 0 \quad \overline{p}^T A(t) \geqq 0$.

Now $\zeta(t)$ cannot be zero; if $\zeta(t) = 0$ then (5.4.19) gives

(5.4.21) $\quad -A(t)\overline{y}(t) = w \quad \overline{y}(t) \geqq 0$

Premultiplying (5.4.21) by $\overline{p}$,

(5.4.22) $\quad -\overline{p}^T A(t)\overline{y}(t) = \overline{p}^T w$

Since $\overline{p}^T A_j(t)\overline{y}^j(t) \geqq 0$ if $\overline{y}^j(t) > 0$ by (5.4.20) we obtain a contradiction from (5.4.22) as its L.H.S. $\leqq 0$ and R.H.S. $= p^T w > 0$. So $\zeta(t) > 0$ implying $\overline{p}^T A(t) \leqq 0$ which is condition (b) of 3.4.2 for equilibria.

Further $\overline{y}^j(t) > 0$ implies $\overline{p}^T A_j(t) \geqq 0$ from (5.4.20). Thus we have,

(5.4.23) $\quad \overline{p}^T A(t)\overline{y}(t) = 0$.

Multiplying (5.4.19) by $\overline{p}^T$ we have

(5.4.24) $\quad \zeta(t)\overline{p}^T d(\overline{p},t) = \overline{p}^T A(t)\overline{y}(t) + \overline{p}^T w$

Using (5.4.23) and Walras law, we have $\zeta(t) = 1$ and (5.4.19) becomes

$$-A(t)\overline{y}(t) + d(\overline{p},t) = w$$

which is condition (a) of 3.4.2 for equilibrium at time t.

The limiting values $p(t)$ and $y(t)$ generated in this remark thus provide equilibrium prices and activity levels of the economy at time t. The next lemma asserts that an accurate enough approximation can be obtained by selecting a fine enough mesh without a passage to the limit. The lemma gives a set of bounds on the degree of approximation obtained in relation to the finite chosen mesh size δ.

5.4.25 Lemma:

Let $a = \max_j \sum_{i,j} |A_j^i(t)|$ and $\bar{a}$ be such that $A(t)y + w(t) \geqq 0$, $y \geqq 0$ implies $\sum_{j=1}^{k} y^j \leqq \bar{a}$ for all $t \in [0,1]$. Let $\tilde{a}$ be such that $|d(p,t)| \leq \tilde{a}$ for all $(p,t) \in S \times [0,1]$. Set $b = a\bar{a}\tilde{a}$. Let $\varepsilon \leq \min_t \{\min_{1 \leq i \leq n} w^i(t)\}$. Let $\tilde{\delta}$ be such that $|d(p,t) - d(p',t)| \leqq \varepsilon$ whenever $|p-p'| \leqq \tilde{\delta}$ for all $t \in [0,1]$. Let $\delta \leqq \min(\tilde{\delta}, \frac{\varepsilon}{a\bar{a}})$.

If G is a triangulation of mesh δ and τ be an n-simplex as in remark 5.4.9, $\bar{p}(t)$ any vector in τ and $\bar{y}(t)$ is generated from λ^i $(i \in \Gamma)$ in (5.4.12) as follows: $\bar{y}(t) \in R_+^k$

$$\bar{y}^j(t) = \begin{cases} \lambda^i(t) & \text{if } j = \ell_i \text{ for some } i \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\bar{p}(t), \bar{y}(t))$ forms the approximate equilibrium prices and activity levels such that,

(a) $\bar{p}(t)^T A_j(t) \leqq \delta a$

(b) $\bar{p}(t)^T A_j(t) \geqq -\delta a$

(c) $|d^i(\bar{p},t) + w^i(t) - A^i(t)\bar{y}(t)| \leqq \frac{\delta b + \varepsilon(\bar{p}(t)^T w(t)) + \tilde{a} + \delta n a)}{\bar{p}(t)^T w(t) - \varepsilon}$

Proof: Given the choices of ε and δ, the continuity of $d(\cdot,t)$ and the construction of $\bar{y}$ the proof follows with some effort. We leave out

the details and refer the reader to Lemma 5.2.5 of [107] for the underlying arguments.

It has to be emphasized that the bounds stated in 5.4.24 is far worse than what occurs in practice.

Now we sketch briefly the steps involved in generating the path $\alpha: [0,1] \to R_+^n \times [0,1]$ from which the equilibrium path is extracted. We will not clutter this discussion with details for handling degenerate values. The following algorithm can be elaborated as 5.2.8 to accomplish this.

5.4.26 Algorithm

Start. Let $(\bar{p}(0), \bar{y}(0))$ be an equilibrium of the initial economy. Let τ_0 be the n-simplex in G containing $\bar{p}(0) = p_0$, and σ_0 be the (n+1)-simplex such that $\tau_0 \subset \sigma_0$. Generate the corresponding $\alpha_0 \in \bar{\tau}_0 \subset R_+^n \times \{0\}$, (see remark 5.4.9). If α_0 is not the unique element of $H^{-1}(w)$ in $R_+^n \times \{0\}$ reconstruct H on $R_+^n \times \{0\}$ such that this is the case. Choose a direction $0 \neq u_0 \in \operatorname{Ker} L_{\bar{\sigma}_0}$, such that u_0 points into $\bar{\sigma}_0$. Then with the triplet $(\alpha_0, \sigma_0, u_0)$ go to iteration 0 of the algorithm.

Iteration r = 0, 1, 2, ...

Generate $\alpha_{r+1} = \alpha_r + \theta_r u_r$ where $\theta_r = \sup(\theta: \alpha_r + \theta u_r \in \sigma_r)$. Record $(\alpha_r, u_r, \theta_r)$. If $\alpha_{r+1} \in R_+^n \times \{1\}$ stop. Otherwise compute $\sigma_{r+1} \in G \sim \{\sigma_r\}$. Let $\bar{\tau}_{r+1}$ be the n-cell common to $\bar{\sigma}_{r+1}$ and $\bar{\sigma}_r$.

Compute $u_{r+1} \in \operatorname{Ker} L_{\bar{\sigma}_{r+1}}$ such that $0 \neq u_{r+1}$ points into $\overline{\bar{\sigma}_{r+1}}$. With the triplet $(\alpha_{r+1}, \sigma_{r+1}, u_{r+1})$ go to iteration (r+1).

5.4.27 Theorem

When G is finite, the algorithm 5.4.25 terminates at a point α_1 in $R_+^n \times \{1\}$ in a finite number of iterations.

Proof. Follows from Assumption 5.4.5 and Lemma 5.4.1. Lemma 5.4.12 rules out "rays" in $H^{-1}(w)$. Assumption 5.4.5 rules out a termination in $(\mathrm{bd}\ \mathbb{R}_+^n) \times [0,1]$. Since by construction α_0 is the unique solution to $Bx = w$ on $\mathbb{R}_+^n \times \{0\}$ the path is guaranteed to terminate in $\mathbb{R}_+^n \times \{1\}$.

At each level t, $(\overline{p}(t),\overline{y}(t))$ can be extracted from $\alpha(t)$ using remark 5.4.9.

5.4.28 Remark:

Again when the dynamics of deformation is not specified a suitable homotopy construction as in remarks 5.2.10 and 5.3.8 would enable the algorithm to compute possible equilibrium paths of smooth transition from the initial to the final economy.

5.4.29 Remark:

Using a refining triangulation $\overline{G}$ of $S \times (0,1]$ to generate $\overline{G}$ a subdivision of $\mathbb{R}_+^n \times [0,1]$ we can implement the algorithm for computing equilibrium paths of evolution of the economy (see remarks 5.2.11, 5.2.12 and 5.3.9) or accurate unknown equilibria of a given economy (see remarks 5.2.13 and 5.3.10 for details).

5.4.30 Remark:

The algorithm of this section yields as a by-product a continuously refining (homotopy) version of Scarf's algorithm [107] as implemented using simplicial subdivision. By artificially specifying H (as in 5.4.8) on $(\mathrm{bd}\ \mathbb{R}_+^n) \times [0,1]$ to be the identity map the analogy to a parametric version of Scarf's algorithm is almost complete. Thus our algorithm could be used to study various problems discussed in [107] in a parametric setting.

5.4.31 Remark

The retrogression of the algorithmic path, again, can be prevented at a price using the same technique as in remarks 5.2.14 and 5.3.11.

5.4.32 Remark

In this final remark we drop assumption 5.4.7 and consider the general case of $w(t)$, $t \in [0,1]$. All that is involved is to adapt the construction of H in 5.4.8 for the case when $\pi(p,t) \leqq 0$. Let $w > 0$ be an arbitrary vector in R^n. For any $(p,t) \in S \times [0,1]$ let $\pi(p,t)$ be as in 5.4.8. Then

$$H(p,t) = \begin{cases} -A_{\ell}(t) & \text{if } \pi(p,t) \geqq 0 \\ d(p,t) - w(t) + w & \text{if } \pi(p,t) \leqq 0 \end{cases}$$

To see the motivation for this construction, it is enough to note that $\sum_{\Delta} \lambda^i(t)$ (see equation 5.4.11) is approximately equal to 1, and in the limit $\zeta(t) = \sum_{\Delta} \lambda^i(t) = 1$ (see 5.4.24). Thus all the arguments of 5.4.9 to 5.4.25 hold in general.

5.4.33 Remark

In part II the algorithm of this chapter will be adapted to various applications. The characteristics of the problem studied will be incorporated in defining H.

PART II

APPLICATIONS

CHAPTER 6

ANALYSIS OF TAX POLICIES IN A DYNAMIC GENERAL EQUILIBRIUM FRAMEWORK

6.1 Overview

In this chapter we devise a framework for analyzing various tax policies in a dynamic general equilibrium framework using the algorithms of Chapter 5. The underlying model is that of 3.4 as modified by the inclusion of a very general tax structure. The algorithms of 5.4 would be used to compute the equilibrium paths of economies under changing tax regimes.

Over the last five years Shoven and Whalley have used a modified version of Scarf's algorithm [107] to analyze tax policies and to evaluate the impact of distortionary taxation of capital income in the U.S. [108-112, 115, 129-130]. The chief advantage of such an algorithmic approach over others in economic policy evaluation is that it does not require any localization or linearization assumptions and therefore offers an appropriate tool for analyzing large nonmarginal changes. In most of the alternative models in economics based on a marginal analysis, parameters such as demand elasticities are only evaluated locally in the current observable economic situation. Those models typically make many linear or local assumptions and are either not completely general in allowing interactions between markets or they deal with "binary economies" (e.g., 2 goods, 2 factors, etc.). The general equilibrium framework used here not only permits the analysis

of large policy changes, but also is not restricted to such binary economies. But so far such algorithmic approaches have always used a comparative static framework. Using the algorithms of 5.4 we devise a dynamic setting for these general equilibrium models for analysing tax policies.

First we construct the underlying model incorporating taxes based on the static models used in [112, 113, and 115]. Then we outline an algorithm to trace the equilibrium path when a general continuous change is made in the tax structure of the economy. Such paths provide a dynamic framework for analyzing issues like the distortionary impact of differential taxation on capital gains [112] and comparison of tax structures which yield the same total revenue [115]. These techniques are applicable to evaluate the impact of policy decisions in general.

6.2 The Model with Taxes

The basic economy considered here is the Walrasian model with production described in 3.4, characterized by (1) a set of market demand functions $d: S \rightarrow R^n$, (2) the vector $w \in R^n_+$ of the economy's initial endowments of all the commodities $\ell = 1, 2, \ldots, n$ and (3) a description of the technological production possibilities through a listing of the activity vectors A_j, $j = 1, 2, \ldots, k$. Let the index set $I = \{1, 2, \ldots, m\}$ denote the consumers.

For convenience first let us consider an economy with only *producer taxes*. The more general model with producer and consumer taxes will be considered in 6.3.

As in [113] we make the tax structure very general -- the tax rate used not be uniform across production sectors. Such discriminatory

factor taxes -- i.e., different tax rates facing different producers on their various inputs and outputs -- provide a convenient framework for studying the impact of distortionary or differential taxation policies. Thus we start out with a general Walrasian model with an arbitrary set of *ad valorem* commodity taxes. The revenue generated from the tax system is dispersed among the individual consumers each of whom is assigned an arbitrary share of the total, or retained by the government for the purchase of goods and services.

As in the earlier model of 3.4 the market demand functions are simply aggregates of individual demands d_i each of which are derived from utility maximization subject to a budget constraint. The budget is given by the value of his initial holdings w_i plus his portion θ_i of the total tax revenue (i.e., the budget is $pw_i + \theta_i\tau$). Using the same arguments of 3.2 and 3.4, the demand functions d are functions of the commodity prices p and the total revenue τ. They are continuous, homogeneous of degree zero in all prices and revenue, and satisfy the Walras law. They can be denoted as $d: R_+^{n+1} \to R^n$ where $d(p,\tau)$ is the vector of demands at prices p and total revenue τ. As before the homogeneousness of d allows us to restrict ourselves to the standard simplex $\tilde{S} = \{(p,\tau) \in R_+^{n+1} \mid \Sigma p^i + \tau = 1\}$. Let us denote the augmented vectors (p,τ) as x. Walras law in this framework states,

(6.2.1) $\quad p^T d(p,\tau) = p^T d(x) = p^T w + \tau.$

For this economy with only producer taxes, p are the prices relevant to the consumer. They are seller's prices for inputs (i.e., net of producer input taxes) and buyers' prices for outputs (i.e., gross of producer output taxes).

Recall the matrix A of 3.4, in which A_j represented the j-th feasible activity for $j = 1, 2, \ldots, k$. The producer tax structure is represented by a matrix T, of the same order as A. T_j^{ℓ} is the tax rate on the ℓ-th commodity when used by the j-th production activity. Thus T_j represents the *ad valorem* tax rates applicable to the inputs and outputs of column A_j. We use the convention that T_j^{ℓ} has the same sign as A_j^{ℓ}. For convenience we define a matrix B such that $B_j^{\ell} = A_j^{\ell} \cdot T_j^{\ell}$ for $\ell = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Thus we can represent our economy with producer taxes as $(d(\cdot), w, A, T)$.

6.2.2 <u>Definition</u>

Now the equilibria of $(d(\cdot), w, A, T)$ are defined as follows. $\overline{x} = (\overline{p}, \overline{\tau}) \in \widetilde{S}$ and $\overline{y} \in R_+^k$ represent equilibrium sets of prices, revenue and activity levels if (1) $d(\overline{p}, \overline{\tau}) = w + A\overline{y}$, i.e., supply is equal to market demand for each commodity ℓ, (2) $\overline{p}^T(A-B)_j \leqq 0$ for all $j = 1, 2, \ldots, k$, with equality if $\overline{y}^j > 0$. Thus profit is maximized at prices $\overline{p}$, as argued in 3.4.2.

At such an equilibrium we further have that the amount of revenue distributed $\overline{\tau}$ equals the revenue generated on the production side of the economy at the equilibrating activity levels $\overline{y}$ when prices $\overline{p}$ prevail. To see this, from condition (1) of 6.2.2 we have

(6.2.3) $\quad \overline{p}^T d(\overline{p}, \overline{\tau}) = \overline{p}^T w + \overline{p}^T A\overline{y}$

By Walras law (6.2.1) and (6.2.3) we have

(6.2.4) $\quad \overline{\tau} = \overline{p}^T A\overline{y}$

From (6.2.4) and (6.2.5) we have, $\overline{\tau} = \overline{p}^T B\overline{y}$, the total revenue generated on the production side of the economy

6.2.6 Problem

Given $(d(\cdot,t), w(t), A(t), T(t))$ the economy at time t, for all $t \in [0,1]$ the problem is to compute an equilibrium path $\phi = (\phi_1, \phi_2) = (\bar{x}, \bar{y}, \phi_2): [0,1] \to \tilde{S} \times R^k_+ \times [0,1]$ of the changing economy. If only the tax regime is changed then we consider the special case $A(t) = A$ and $w(t) = w$ for all $t \in [0,1]$.

To compute the equilibrium path we use the algorithm of 5.4, as implemented on a triangulation G of $\tilde{S} \times [0,1]$ and a corresponding subdivision $\bar{G}$ of $R^{n+1}_+ \times [0,1]$ generated by G. From the algorithmic path using Remark 5.4.9.

The details of implementation of the algorithm and interpretation of the algorithmic output are as in 5.4. But the piecewise affine homotopy $H: R^{n+1}_+ \times [0,1] \to R^{n+1}$ and the regular value $\tilde{w} \in R^{n+1}$ (such that $H^{-1}(\tilde{w})$ contains the algorithmic path $\alpha: [0,1] \to R^{n+1}_+ \times [0,1]$) has to be constructed with care incorporating the characteristics of the present model. Such a construction follows.

6.2.7 Construction of H

Define $B^\ell_j(t) = A^\ell_j(t) T^\ell_j(t)$, for $\ell = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$ and $t \in [0,1]$. As in 5.4.8 we specify H on the vertices of the triangulation G of $\tilde{S} \times [0,1]$ and then construct its piecewise affine extension to $R^{n+1}_+ \times [0,1]$. For each point $(x,t) = (p,\tau,t) \in G^0$, set $\pi(x,t) = \max\limits_{1 \le j \le k} p^T(A(t) - B(t))_j$ the maximum per unit after tax profit among all the activities of $A(t)$ when the prevailing prices are p. Let $r(x,t) = r$ be an index such that $\pi(x,t) = p^T(A(t) - B(t))_r$. Now define, for $(x,t) = (p,\tau,t) \in G^0$, $H(x,t) \in R^{n+1}$ as follows

$$H(x,t) = \begin{cases} \begin{bmatrix} -A_r(t) \\ p^T A_r(t) \end{bmatrix} & \text{if [illegible]} \\ \begin{bmatrix} d(x,t) - w(t) + v \\ -\tau + \Delta \end{bmatrix} & \text{if [illegible]} \end{cases}$$

where $v > 0$ is an arbitrary vector in R^n, $0 < \Delta < 1$. Define $w = \begin{bmatrix} v \\ \Delta \end{bmatrix} \in R_+^{n+1}$. Perturb w to the regular value $\bar{w}$. $H^{-1}(\bar{w})$ contains the algorithmic path.

6.2.8 Remark

The construction of the algorithmic path and the extraction of the equilibrium path is done exactly as in 5.4.9. To show that the algorithmic path $\pi: [0,1] \to R_+^{n+1} \times [0,1]$ constructed using Algorithm 5.4.26 in $H^{-1}(\bar{w})$ approximates the required equilibrium path we can use the same arguments as in Remark 5.4.9. For details of the underlying logic in the static version see [113].

6.2.9 Remark

Since $T(t)$ the tax structure at time t is general enough to incorporate preferential taxation of specific commodities or specific activities our framework is very suitable to study the impact of distortionary preferential taxation. Let $\beta \subset \{1, 2, \ldots, k\}$ index activities of a particular industrial sector under study. Let $\gamma \subset \{1, 2, \ldots, n\}$ index a subset of the commodities subject to special taxes. Let $\bar{\beta} = \{1, 2, \ldots, k\} \sim \beta$ and $\bar{\gamma} = \{1, 2, \ldots, n\} \sim \gamma$. Defining $T(t) = [T_\beta(t) \;\; T_{\bar{\beta}}]$ would define a taxation policy where the sector β is under a special taxation study. Similarly $T(t) = \begin{bmatrix} T^\gamma(t) \\ T^{\bar{\gamma}} \end{bmatrix}$

represents a taxation policy where distortionary taxation of certain commodities indexed by γ is under study. If commodities γ are subject to special taxation policy in sector β, then tax structure $T(t)$ representing the preferential taxation changes can be denoted by

$$T(t) = \begin{bmatrix} T_{\beta}^{\gamma}(t) & T_{\bar{\beta}}^{\gamma} \\ T_{\beta}^{\bar{\gamma}} & T_{\bar{\beta}}^{\bar{\gamma}} \end{bmatrix}.$$

Thus the equilibrium paths under various types of continuous changes in the tax regime can be studied by our algorithms.

6.2.10 Example

As discussed in its static version in [110, 112] the above framework lends itself to a straightforward application to study the incidence effects and the efficiency costs associated with the differential taxation of income from capital in the U.S. economy. We consider the following illustrative example in which the existing regime with differential taxation across sectors of income accruing to capital, is slowly changed to a tax regime of uniform taxation.

$$A = \begin{array}{c} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1.4 & -1 & -.5 & -.86 \\ 0 & 0 & 0 & -1 & -0.7 & -1 & -.6 & -.42 \end{bmatrix} \end{array} \begin{array}{l} \text{output 1} \\ \text{output 2} \\ \text{labor} \\ \text{capital} \end{array}$$

$\beta_1 = \{5, 6\}$ indexes a heavily taxed sector.

$\beta_2 = \{7, 8\}$ indexes a lightly taxed sector.

$\gamma = \{4\}$ indexes 'capital' which is the differentially taxed commodity. The tax structure at time $t = 0$ is as follows.

$$T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & .04 & .04 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .04 & -.04 \\ 0 & 0 & 0 & 0 & -.04 & -.04 & -.04 & -.04 \\ 0 & 0 & 0 & 0 & -.15 & -.16 & -.05 & -.04 \end{bmatrix}$$

Sector 1 Sector 2

$$T(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & .04 & .04 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .04 & .04 \\ 0 & 0 & 0 & 0 & -.04 & -.04 & -.04 & -.04 \\ 0 & 0 & 0 & 0 & -.04 & -.04 & -.05 & -.04 \end{bmatrix}$$

$T_1^v(t) = [-.15 \;\; -.16] + t^2[.11 \;\; .12]$ is the submatrix of $T(t)$ denoting a particular dynamic tax reform path, at $t = 1$ the tax structure taxes capital on both the sectors uniformly

The equilibrium path of the economy under the changing tax regime can be computed using our algorithm of 5.4.

6.2.11 Remark

To compare one equilibrium of an economy to another we need a criterion function based on measures of welfare and optimality. Even though no such generally accepted criteria exists let us assume the existence of such a measure $\psi: E \to \mathbb{R}$ where E is the set of all general equilibria. In studying the effects of distortionary taxation measures of efficiency losses have been attempted based on a form of welfare analysis in the tradition Marshallian consumer surplus, e.g., Harberger [51-53]. We will not pursue the details of formulating such measures of comparing equilibria.

We further assume that we can compare equilibrium paths of transition between two economies by a functional $\Lambda: P \to R$ where P is the set of all possible equilibrium paths between the initial and the final economy. The functions ψ and Λ enable us to talk about comparing equilibria and equilibrium paths of economies.

6.3. Extension to Consumer Taxes

Extending the current model to include consumer taxes is straightforward. Such an extension is extremely useful allowing consideration of a wide range of fiscal phenomena like the operation of statewide taxes or subsidies in a national economy, citywide sales taxes in a state economy and other issues directly linked to the consumer.

Each consumer $i \in I$ faces a vector of finite *ad valorem* consumption tax rates $c_i \in R^n$ payable on purchase of all the goods. Note the generality of the differential tax structure allows taxes or subsidies on selected commodities for selected consumers in the economy. For convenience we define a vector ξ_i, such that $\xi_i^\ell = c_i^\ell \cdot p_i^\ell$. Each individual i faces an effective price vector $p + \xi_i$, and pays consumer taxes $\phi_i(p,\tau) = \xi_i^T d_i(p,\tau)$. τ is again the total revenue of which he receives a portion θ_i.

Let $\phi(p,\tau) = \sum_{i=1}^{m} \theta_i(p,\tau)$. Then Walras law for this case is, for all $p \in S$:

(6.3.1) $\quad p^T d(p,\tau) + \phi(p,\tau) = p^T w + \tau.$

A general equilibrium can be characterized as in 6.2.2. At the equilibrium value of $(\bar{x},\bar{y}) = (\bar{p},\bar{\tau},\bar{y})$ we have

(6.3.2) $\quad \bar{\tau} = \bar{p}^T B\bar{y} + \phi(\bar{p},\bar{\tau}).$

6.3.3 Construction of H

In this model the algorithmic details are identical to that in 6.2 except for a minor difference in the construction of H

$$H(x,t) = \begin{cases} \begin{bmatrix} -A_r(t) \\ p^T A_r(t) \end{bmatrix} & \text{if } \pi(x,t) > 0 \\ \begin{bmatrix} d(x,t) - w(t) + v \\ -\tau + \phi(x,t) + \Delta \end{bmatrix} & \text{if } \pi(x,t) \leq 0 \end{cases}$$

where $\pi(x,t)$, v and Δ are as in 6.2.4.

6.3.4 Remark.

A further modification in the above model is to incorporate the fact that when different tax systems are being compared one often wants to ensure that the different systems generate the same real yield. Equilibrium paths of economies under changing tax structures which generate the same real yield could be studied using our algorithms. Same aspects of such a model in the static framework have been considered by Shoven and Whalley [115].

6.3.5 Remark.

The underlying model and the tax structure incorporated seems rich enough for many other promising applications. One of the advantages of this type of technique is that more than one distortion or policy change can be simultaneously modeled. As indicated in [111] a natural extension will involve embedding the capital income tax structure into the entire U.S. tax system incorporating such things as the federal income tax and the social security system. Integration of the corporation and personal income tax systems, introducing a negative income tax,

financing social security out of increased income taxes or instituting a value added tax as a partial replacement for property taxes, etc., are policy changes which could be evaluated. The impact of labor unions or wage subsidy programs would also be studied in variations of the same basic framework.

6.4 Equilibrium Paths of Public Goods Economies under Deformation

Another direct application of our algorithm of 5.4 is to compute an equilibrium path for a continuously changing *Public Goods Economy*. For each period $t \in [0,1]$ the algorithmic path would approximate elements of a certain subset of Pareto optimal allocations in the public goods economy for that period.

6.4.1 Basic Structure

A brief outline of the basic structure of these economies are as follows. Consumers are partitioned into a number of exogenous public (governmental) jurisdictions, which provide public goods and collect taxes. All members (and only members) of a given jurisdiction consume the total of the public goods provided by the jurisdiction. Some kind of proportional wealth tax is levied on its members by each jurisdiction to raise sufficient revenue to cover the cost of public goods provided. Private goods are traded across jurisdictions and production is not jurisdiction specific. See also [97, 99].

6.4.2 Equilibrium paths

Our algorithm of 5.4 will be able to compute approximate equilibrium paths ϕ with the following characteristics. Given the public goods economy at time $t \in [0,1]$, $\phi(t)$ would approximate elements of a certain subset of Pareto optimal allocations which are consistent

with profit maximization on the part of the producers, and utility maximization over private goods bundles subject to after-tax budget constraints by consumers. Such Pareto optimal allocations will be consistent with decentralized decision making and correspond to rather realistic tax structures.

6.4.3 Strategy:

Richter [97, 99] has applied Scarf's algorithm [107] to compute approximate equilibria of public goods economies in a static framework. Using a procedure identical to that adopted in 6.2 and 6.3 we can cast the method of [99] in a dynamic framework and implement the dynamic version using the algorithm of 5.4. The algorithmic output, interpreted using 5.4.9, will yield the required equilibrium path.

6.4.4 Remark

As shown in [98] the existence of equilibria for the public goods economies can be demonstrated using Kakutani's fixed point theorem. Thus the algorithm of 5.3 can also be used to compute approximate equilibrium paths of these economies under continuous deformation.

CHAPTER 7

ISSUES IN INTERNATIONAL TRADE

7.1 Model of International Trade

In this chapter we extend the framework we have developed to trace equilibrium paths in international markets under continuous deformation due to changes in tariff structure, import quotas, formation or expansion of customs unions, etc. Such a procedure provides a dynamic framework for evaluating the impact of tariff reforms, fiscal harmonization in common markets, gains and losses from the formation and expansion of customs unions and other issues in world trade.

Our model is a dynamic version of the basic model used in [114, 54]. There are n commodities where a commodity is thought of as a particular good in a particular country. In other words similar goods in different countries are treated as different commodities. The consequent large dimensionality poses a computational burden; alternate formulations, as for example, based on [54] will alleviate this problem. In the following presentation of the basic model we shall not be concerned about such issues. See Remark 7.2.5 also.

$p \in R_+^u$ is the world price system. These prices are net of any tariffs and may be thought of as seller's prices.

$\tilde{Q} = \{1, 2, \ldots, q, \ldots, Q\}$ indexes the finite number of countries. Each country q has, associated with it, a set Y_q of 'domestic' commodities which are produced or initially located in country q. Y_q, $q \in \tilde{Q}$ partition $\{1, 2, \ldots, u\}$ such that $|Y_q| = n_q$, $u = \sum_{q=1}^{Q} n_q$

Each country imposes a vector of *ad valorem* tariff rates $T_q \in R^n_+$ which are paid by the consumers of the country q. $T^q_q = 0$, i.e., tariff rates applying to domestic commodities are zero. The purchase prices (gross of tariffs) faced by country q are given by $\tilde{p}_q = p + g_q$ where $g_q \in R^n$ is defined as $g^i_q = p^i \cdot T^i_q$. τ_q the tariff revenue received by the q-th government is distributed among the consumers of that country. As before the total demands of country q are assumed to be functions of $\tilde{p}_q$ and τ_q, $d_q: R^{n+1}_+ \to R^n$ are continuous. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_Q) \in R^Q$. Then the total world demands, $d: R^n_+ \times R^Q \to R^n$ are continuous, $d(p,\tau)$ is homogeneous of degree zero in (p,τ). For country q the sum of the budget constraints implies

$$(7.1.1) \quad \tilde{p}_q^T d_q(\tilde{p}_q,\tau_q) = p^T w_q + \tau_q$$

where $w_q \in R^n$ is the total endowment of the q-th country. $w = \sum_{q=1}^{Q} w_q$ is the world's initial endowments. Let $\phi_q(p,\tau_q) = g_q^T d_q(\tilde{p}_q,\tau_q)$ be the tariff revenue collected by country q. From (7.1.1) we have,

$$p^T d_q(\tilde{p}_q,\tau_q) + \phi_q(p,\tau_q) = p^T w_q + \tau_q.$$ Summing over q,

$$(7.1.2) \quad p^T d(p,\tau) + \sum_{q=1}^{Q} \phi_q(p,\tau_q) = p^T w + \sum_{q=1}^{Q} \tau_q.$$

All the production possibilities in the world are represented by $A \in R^{n\times k}$, where A_j denotes the j-th activity.

7.1.3 <u>Assumptions</u>

As usual the set $\{y \in R^k_+, Ay + w \geq 0\}$ is bounded. Further it is assumed that each activity A_j uses only commodities located in one country. Thus A can be partitioned into a matrix of free disposal activities and a block diagonal sequence of arrays.

7.1.4 Definition

The international economy described above can be represented by (d,w,A,T) where d, $w \in R^n$ are the aggregate demand function and total endowment of the world, A is the technology matrix specifying the production possibilities, $T \in R^{n \times Q}$ be the tariff structure prevailing, i.e., T_q denotes the tariffs faced by the q-th country.

7.1.5 Equilibrium

Given the economy (d,w,A,T) as defined above, $\overline{x} = (\overline{p},\overline{\tau}) \in R_+^{n+Q}$, $\overline{y} \in R_+^k$ represent an *international tariff general equilibrium* if

(i) demand and supply balance for each commodity,

i.e., $A\overline{y} + w = d(\overline{p},\overline{\tau})$

(ii) profit is maximized at prices $\overline{p}$, $\overline{p}^T A_j \leq 0$, $j = 1, 2, \ldots, k$, with equality if $\overline{y}^j > 0$ (as in 3.4.2)

(iii) the tax revenue received by each country equals that dispersed by the government of that country, i.e.,

$$\sum_{i=1}^{n} \overline{p}^i \, T_q^{\,i} \, d_q^{\,i} \, (\overline{p}_q,\overline{\tau}_q) = \overline{\tau}_q$$

7.2 Implementation in the Dynamic Framework

7.2.1 Problem

Given $(d(\cdot,t), w(t), A(t), T(t))$ the economy at time t, for all $t \in [0,1]$ the problem is to compute a path of approximate international tariff equilibria, i.e., $\phi = (\phi_1,\phi_2) = (x,y,\phi_q) : [0,1] \to \hat{S} \times R_+^k \times [0,1]$ of the changing economy. $\hat{S}$ is the standard simplex in R^{n+Q}. Recall $\overline{x} = (\overline{p},\overline{\tau})$ where $\overline{p} \in R_+^n$, $\overline{\tau} \in R_+^Q$. If only the tariff structure is altered then we consider the special case $A(t) = A$ and $w(t) = w$ for all $t \in [0,1]$.

To compute the equilibrium path required we use the algorithm of 5.4, as implemented on a triangulation $\hat{G}$ of $\hat{S} \times [0,1]$ and a corresponding subdivision $\overline{G}$ of $R_+^{n+Q} \times [0,1]$ generated by G. From the algorithmic path α $[0,1] \to R_+^{n+1} \times [0,1]$ we generate the required equilibrium path using Remark 5.4.9. The details of implementation and interpretation of the algorithmic output are as in 5.4. But the piecewise affine homotopy, H $R_+^{n+Q} \times [0,1] \to R^{n+Q}$ and the regular value $\bar{w} \in R^{n+q}$ (such that $H^{-1}(\bar{w})$ contains α the algorithmic path) has to be constructed suitably.

7.2.2 Construction of H

Let G be a triangulation of $\hat{S} \times [0,1]$. As in 5.4.8 we specify H on the vertices G^0 and then construct its piecewise affine extention to $R_+^{n+Q} \times [0,1]$. For each point $(x,t) = (p,\tau,t) \in G^0$, set $\pi(x,t) = \max_{1 \leq j \leq k} p^T A_j(t)$ the maximum profit from all the activities. Let $r(x,t) = r$ be an index such that $\pi(x,t) = p^T A_r(t)$. Now define for $(x,t) \in G^0$,

$$H(x,t) = \begin{cases} \begin{bmatrix} -A_r(t) \\ 0 \end{bmatrix} \in R^{n+Q} & \text{if } \pi(x,t) > 0 \\ \begin{bmatrix} d(x,t) - w(t) + v \\ -\tau + \Delta e \end{bmatrix} & \text{if } \pi(x,t) > 0, \end{cases}$$

where $v > 0$ is an arbitrary vector in R^n, $\tau \in R^Q$ as defined earlier, $e \in R^Q$ the vector of 1's and $0 < \Delta < 1$.

Define $w = \begin{bmatrix} v \\ \Delta e \end{bmatrix} \in R^{n+Q}$. Perturb w if necessary to the regular value $\tilde{w}$.

7.2.3 Remark

The construction of the algorithmic path and the extraction of the equilibrium path is done as in 5.4.26 and 5.4.9. To show that the algorithmic path α constructed using Algorithm 5.4.26 in $H^{-1}(\bar{w})$ approximates the equilibrium path we can proceed as in Remark 5.4.3. The parallel arguments for the static model are given in [114].

7.2.4 Remark

The generality of the tariff structure T(t) gives us flexibility for implementing selective tariff reforms as applied to a subset of the commodities in selected countries. Features like import quotas can be easily built into our framework and studied. Whalley [129] examines the process of harmonization to a common output tax system among EEC member states within a static general equilibrium framework. For each of the economies in the EEC the tax system is replaced by each of the tax systems prevailing in other member states and the new competitive equilibria are computed. Applications of this type where a process of change is examined can be more effectively studied in a dynamic setting like ours.

Equilibrium paths of international economies under deformation due to changes in inflation rates, exchange rates, production technologies, etc., can be studied in our framework.

7.2.5 Remark

Helpman [54] has improvised Scarf's algorithm [107, p. 123] to compute equilibria in international markets (in a static framework) which alleviates the problem of dimensionality resulting from the redefinition of each good or factor in each country as a distinct good or factor. His approach necessitates simplicial pivoting only on

$\bar{S}$ the standard simplex in the space of primary factor prices (whose dimension is independent of the number of goods) instead of $\hat{S}$ used in 7.2.1. But the approach of [54] could be cast in a dynamic framework in a straightforward manner as in the various applications of Chapters 6 and 7. The techniques and algorithm of 5.4 can be implemented with the reduced dimensionality feature to compute an approximate path of world trade equilibria.

CHAPTER 8

DYNAMIC ANALYSIS OF SPATIAL EQUILIBRIUM MODELS

8.1 Overview

In this chapter the algorithms of Chapter 5 are applied to analyze some spatial equilibrium models under deformation based on computed paths of general equilibria. As examples some urban equilibrium models and a wide class of spatial equilibrium models are discussed. The algorithm of 5.2 or variations of it are applicable to the urban models while the algorithm of 5.3 can be adapted for the class of spatial equilibrium models of 8.4. During the last five years these models have been studied in a static framework (for example, see [4-5, 79-81]) using simplicial pivoting algorithms. These prior studies have emphasized the need for a dynamic framework in the area.

In the discussion below our main emphasis would be on the adaptation of our algorithms for a dynamic analysis of the issues concerned. We will not go into the details of the underlying models and the numerous issues which can be analyzed using our framework. Many of these possibilities are discussed in [4-5, 79-81].

8.2 Discrete General Equilibrium Urban Models

The salient features of an urban model from residential location theory is given below in an abstract form. The emphasis here is on the adaptability of the algorithm of 5.2 to compute the equilibrium paths of these models under deformation induced by the urban change studied.

Depending on the specific urban issue of concern (for example urban transportation changes [4], property taxes [5] and congestion [79]) the corresponding sector of the following model can be elaborated to incorporate the necessary detail.

8.2.1 The model:

The land area of the prospective city (all of which might not be put to residential use) is subdivided into n sections, $i = 1, 2, \ldots, n$. The central business district (CBD) is located at a point of effective distance d_i from land section i. The residents of the city are indexed by $r = 1, 2, \ldots, R$. Every resident is required to commute back and forth to the CBD, the r-th resident, m_r times a day. Let t_i be the time of a round trip from i-th section to CBD. p_i is the unit price of land in section i. The only nonland good in the model has a price p_0. Thus $p \in R^{n+1}$ represents the existing price system.

The welfare of the residents depend on consumption of housing h, consumption of nonland good $\tilde{x}$ (net of $\overline{x}$, which is used as input into the production of housing services) and effective leisure time $\tilde{T}$ (net of commuting time $\overline{T}$). Thus if x is the total nonland good consumed and τ is the total time available after work, sleep and other necessary activities, we have

(8.2.2) $x = \overline{x} + \tilde{x}$ and

(8.2.3) $t = \overline{T} + \tilde{T}$

The utility function of the r-th resident U^r is described as

(8.2.4) $U^r = U^r(h, \tilde{x}, \tilde{T})$.

Housing services are produced according to the production function h given as,

(8.2.5) $h = h(\hat{x}, \ell)$ where ℓ is the amount of land used.

As is customary in urban models land ownership is assumed to be exogenous, i.e., the residents do not have land as initial endowment. The initial endowment of the r-th resident is exclusively nonland good, say w_r units.

The maximization problem which faces each consumer is involved. First he must decide how much land, housing and goods he would buy if he were forced to live in section i, i = 1, 2, ..., n. Then he must decide on what section to live in.

For each section i, r-th consumer maximizes his utility (8.2.4) subject to (8.2.5) and the budget constraint,

(8.2.6) $\quad p_i \ell_i + p_0 (x + E_i) = p_0 w_r$,

where ℓ_i is the quantity of land in section i purchased and E_i is the commuting expenses applicable in section i. If $\overline{U}_i^r$ is the optimal value of the objective thus obtained, then his final demand choices are made based on the optimization over all the sections,

(8.2.7) $\quad \overline{U}_q^r = \max\limits_{1 \le i \le n} \overline{U}_i^r$.

Let ℓ_q^r and x^r be the corresponding demand for land and nonland goods. They constitute components of his demand vector $d_r \in R^{n+1}$ at the prevailing prices p; $d = \sum_{r=1}^{R} d_r$ provides the aggregate demand at prices p. $d: R_+^{n+1} \to R^{n+1}$ the demand function is homogeneous of degree zero in prices. As usual, we restrict the prices to the standard simplex $\tilde{S} = \{p \mid \sum_{i=1}^{n} p_i = 1,\ p \geq 0\}$. If $w \in R^{n+1}$ represents the vector of total initial endowments of the economy (the first component that of nonland goods and the rest n components the total land in each of the n sections) then we have the excess demand function $g: S \to R^{n+1}$ defined as $g(p) = d(p) - w$.

Equilibria can be characterized as prices at which there is no excess demand for land or nonland goods, i.e., $\bar{p}$ such that $g(\bar{p}) \leq 0$.

8.2.8 Remark:

Let us assume that g is continuous. Then the techniques and algorithms of 5.2 can be applied to this urban model. Given that $g(\cdot,t): s \to R^{n+1}$ represents the excess demands at time $t \in [0,1]$ we can compute an approximate equilibrium path of prices for land and nonland goods. The details are as given in 5.2. Deformation in the economy is modeled as being caused by changes in the particular aspects of the urban structure under study. The possibilities of issues which could be studied in such a framework are endless; some of them will be specified in 8.3.

8.2.9 Remark:

The earlier work in the area has been in a comparative static framework [4-5, 79]. A particular strategy for simplicial pivoting called *Sandwich method* was used with integer labeling [73]. Our algorithm subsumes such methods; using the equivalence of vector and scalar labels in simplicial approximation (see for example [44]) we can incorporate alternate scalar or vector labels in constructing H in 5.2.4 and implement the algorithm of 5.2.

8.2.10 Remark:

To make the size of the city to be determined in the model the following device can be used. Let the land have a nonurban use (say, agriculture) such that when so used each unit of land in any section i can produce g units of nonland goods. Thus an additional constraint is operative in maximizing (8.2.4),

(8.2.11) $\ell_i^r = 0$ if $p_i < p_0 g$ for all i, r

Thus land will not be used for urban purposes unless it can command a price of at least $p_0 R$.

8.3 Dynamic Modeling of Urban Problems

In this section, modeling of certain specific urban problems in the framework of 8.2 will be briefly indicated. The strategy is to emphasize and elaborate the relevant sector of the model incorporating all the required detail. We follow [4-5, 79].

8.3.1 Transportation changes

To study effects of changes in transportation (for example, government policies, altering prices or upgrading facilities) the modal choice in transportation, its resulting impact on effective leisure time, transportation expenditure, etc., has to be modeled in greater detail. Let the modes of transportation available be indexed by $k = 1, 2, \ldots, K$. Then $\bar{T}$ in (8.2.4) is a function of the modal choice k, in addition to d_i and w_r as in 8.2.1. Similarly E_i in (8.2.6) is also a function of the modal choice k, d_i and w_r. Changes in the expenses and quality of various modes of transportation induces changes in the equilibrium prices, size of the city, welfare levels, population density, etc. For details of such a study in a static framework, see [4].

8.3.2 Property taxes

To study the effects of changing property taxes, the supply side of the housing market is modeled in detail. Reformulate h in (8.2.5) as a function of land, structure and the nonland good, i.e., $h(\bar{x}, \ell, \theta)$ 8, the structure can be measured in terms of living area, floor space of standardized quality. The budget constraint of a resident is

reformulated including the relevant property taxes. (8.2.6) is replaced by,

(8.3.3) $\quad p_0(\bar{x} + E_i) + c_h(1 + \rho)h = p_q w_r(1 - \delta)$

where c_h is the cost of unit housing in section i (including the price of land), δ is the income tax rate and ρ the property tax rate. The effects of long run changes in property taxes can be stated using our algorithms. For details of an implementation in the static framework, see [5].

Various problems in urban economics like congestion, effects of racial discrimination and segregation in housing markets, effects of changes in transportation systems, etc., can be modeled and studied in a dynamic framework using our algorithms on more elaborate versions of the model in 8.2 (see [79]).

8.3.3 Remark

Deschamps [19] has used a fixed point algorithm to the related problem of pricing for congestion in telephone networks. The problem of simultaneously determining a vector of optimal qualities of service as well as prices and capacities for a telephone utility can be formulated as that of computing Kakutani fixed points of an appropriate U.S.C. mapping (see [19] for details). To compute paths of approximate equilibria for such systems the algorithm of 5.3 can be applied. The technique promises to be applicable in general to the pricing of public utilities.

8.4 Spatial Equilibrium Models

In this section we indicate the use of algorithm 5.3 to a wide variety of models which arise in spatial economics. The class of

spatial equilibrium models discussed here include as special cases the generalized Von Thünen models [81] and the generalized Transportation problem [80].

In the following characterization of the essential features of the class of spatial equilibrium models under reference, we follow MacKinnon [80]. There are n geographically separated markets indexed by $\ell \in L$ where the ℓ-th market trades in one good at price p^{ℓ}. There may be more than one market at a given point provided they trade in different goods. The same good may be traded at more than one market provided that the markets are located at different points. $p \in R_{+}^{n}$ is the vector of market prices.

8.4.1 Assumptions

The demand and supply functions are U.S.C. point-to-set maps with the following properties:

(1) $D_{\ell}, S_{\ell}\ R_{+}^{n} \to C(R_{+})$ for $\ell \in L$ Where $C(R_{+})$ is the collection of convex subsets of R_{+}.

(2) When $p^{\ell} \leq 0$ then $0 \in S_{\ell}(p)$, when $p^{\ell} < 0$ then $S_{\ell}(p) = \{0\}$.

(3) There exists a scalar c, $0 < c < \infty$ such that whenever $\Sigma p^{\ell} \geq c$ we have $\{\sum_{\ell=1}^{n} D_{\ell}(p) - S_{\ell}(p)\} \cap R_{+} = \phi$

For motivation for these assumptions see [81]

Apart from the markets discussed the model has m *trading sites* (indexed by $j \in J$) which are subsidiary institutions located at different geographical sites. The prices operative at trading sites are functions of p the vector of market prices. The exact nature and role of trading sites differs greatly among models.

The demand D_ℓ associated with ℓ th market is the sum of demands of all the trading sites for the good traded at market ℓ, i.e.,
$D_\ell(p) = \sum_{j=1}^{m} D^j(p)$ Similarly $S(p) = \sum_{j=1}^{m} S^j_\ell(p)$

8.4.2 Definition:

An equilibrium of the spatial equilibrium model is defined as a set of prices p, supplied from trading site j sold at market ℓ, s^j_ℓ and demands from trading site j satisfied at market ℓ, d^j_ℓ, such that,

$s^j_\ell \in S^j_\ell(p)$ $j \in J$, $\ell \in L$

$d^j_\ell \in D^j_\ell(p)$ $j \in J$, $\ell \in L$.

$\sum_{j=1} d^j_\ell = \sum_{j=1} s^j_\ell$ $\ell \in L$.

8.4.3 Problem

Given the spatial economy $(S^\ell_j(\cdot,t), D^\ell_j(\cdot,t), \ell \in L, j \in J)$ for all $t \in [0,1]$ an approximate equilibrium path can be computed using the algorithm of 5.3. The strategy is to define the economy as a point-to-set map defined on $\hat{S} \times [0,1]$. Then a suitable homotopy H is constructed such that a route traced in $H^{-1}(0)$ using the algorithm of 5.3 yields the required path.

Let $\bar{S} = \{x \in R^{n+1}_+ \ \ \Sigma x^\ell = 1\}$ Define $\phi\ S \mapsto R^n_+$ as follows

$\phi^\ell(x) = x^\ell / x^{n+1}$ $x^{n+1} > 1/x$

$\phi^\ell(x) = \dfrac{x^\ell c}{(1-x^{n+1})}$ otherwise.

8.4.4 Definition of H

Let G be a triangulation of $\tilde{S} \times (0,1]$. H is defined on G^0 as follows. For $(x,t) \in G^0$, $H^\ell(x,t) = h^\ell(x,t)$ for $\ell = 1, 2, \ldots, m$ where $h^\ell(x,t) \in D_\ell(\phi(x),t) - S_\ell(\phi,(x),t)$. $H^{n+1}(x,t) = \sum_{\ell=1}^{n} H^\ell(x,t)$

Now extend H as usual to all of $\bar{S} \times [0,1]$ as the piecewise affine map,

$$H: \bar{S} \times [0,1] \to \mathrm{Aff}(\tilde{S} - \tilde{S}).$$

Now using a algorithm 5.3.7 we can generate the required path of approximate equilibria. For details of the implementation, see 5.3. For parallel arguments in the static framework using a *vector sandwich* method, see [81].

8.4.5 <u>Remark</u>

The models of this section form a very general class of spatial equilibrium models. They include the generalized transportation models [80], generalized Von Thünen models of land use [81], etc., as subclasses.

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BIOGRAPHICAL SKETCH

Kose John was born in Chowghat, Kerala (India) on March 27, 1951. After secondary schooling at Model High School and Predegree education at Intermediate College in Trivandrum, India, he joined an accelerated undergraduate program in Physics -- B.Sc. Special -- at University College, University of Kerala, Trivandrum. Having been selected as a National Science Talent Scholar (1967-72) he participated in various national research programs in Physics. One of the projects, undertaken at Indian Institute of Technology, Kanpur, studied "Electrical resistivity of Cd^{+} at He temperatures".

In 1972 he came to the United States and joined the Master's Program in Computer Sciences at Florida Institute of Technology, Melbourne. As a part of their adjunct faculty he taught courses in differential equations, calculus and physics. In 1974 he accepted a Graduate Council Fellowship at the University of Florida to pursue his doctorate in business administration with a major in management. He has been teaching senior level courses in Managerial Operations Analysis over the last two years, he received the outstanding graduate teaching award recently. His areas of active interest outside mathematical programming (complementary pivot theory) are mathematical economics and theory of finance. He has accepted a position as an assistant professor in the Graduate School of Business, New York University, New York.

He has edited two journals of popular science, Sciencevei and Sophik in India. Another extra curricular interest has been Debating; he has won several prizes in national level inter-university competitions. The team he led from F.I.T., Melbourne, to the World Youth United Nations session at New York in 1973 won the award for the best delegation. His hobbies include swimming and photography.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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